Chapter 4
Model-Based Measurement of Name Concentration Risk in Credit Portfolios

4.1 Fundamentals and Research Questions on Name Concentration Risk

As described in Sect. 2.6, name concentration risk arises if the idiosyncratic risk cannot be diversified away, which concurrently means that assumption (A) of the ASRF model, the infinite granularity, does not hold. However, a violation of (A) does not have to lead to the fact that the ASRF framework cannot be used at all for credit risk quantification. Nonetheless, the consequences of the violation have to be considered, i.e. the existence of name concentration risk. This issue is not only a problem that should be accounted for in credit risk management when dealing with analytical models, but it is also critical for supervisory capital measurement in banks.[162] This raises the following question: Does assumption (A) of the IRB-model under Pillar 1 generally hold for our portfolio or do we have to quantify name concentration risk for Pillar 2?

Emmer and Tasche (2005) show that the underestimation of *individual name concentrations* can have a significant impact, especially if the exposure weight of a single credit is higher than 2%. Due to the limits on large exposures in the European Union, the exposure to a client may not exceed 25% of a credit institution's own funds.[163] Consequently, a weight of 2% (of total funds) can only be exceeded if (1) more than 8% of a credit institution's capital are own funds and (2) the large exposure limit is reached. This shows that idiosyncratic name concentrations usually should not be problematic if the large exposure rules are effective. Similarly it could be quantified whether *portfolio name concentration* has a significant impact on the risk of the portfolio. In this context, it would be interesting to know which characteristics a real-world bank portfolio should fulfill in order to get a sufficient approximation

[162]Another solution to the problem of the violation of assumption (A) or (B) might be to cancel risk quantification under the IRB Approach and use internal models. However, this solution is not designated in Basel II.
[163]Cf. Sect. 3.2.

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of the "true" risk even if name concentrations are not explicitly measured. These characteristics should be determined in a way that the accuracy of the ASRF framework can be easily assessed for a broad range of credit portfolios. If the desired accuracy cannot be achieved using the ASRF model, the VaR of the portfolio could be approximated using the granularity adjustment formula. However, since this formula does not provide an exact solution but an approximation of the risk stemming from portfolio name concentration, it is important to know for which types of credit portfolios the adjustment formula shows an adequate performance. Unfortunately, the existing literature concerning name concentration risk does not answer these questions sufficiently.[164] Against this background, the following important tasks regarding name concentrations will be analyzed in this chapter:

- In which cases are the assumptions of the ASRF framework model critical concerning the credit portfolio size?
- In which cases are currently discussed adjustments for the VaR-measurement able to overcome the shortcomings of the ASRF model?

The answers to both questions would be available if the minimum number of loans, which is necessary to fulfill the granularity assumption (A) with a required accuracy, were known. For this purpose, it could be demanded that the analytically determined VaR and the true VaR using the binomial model of Vasicek shall differ at maximum 5%.[165] Against this background, firstly, the formulas for the (first-order) granularity adjustment will be derived.[166] As the granularity adjustment itself is an asymptotic result, it can be seen as an approximation for medium grained portfolios. Thus, the existent framework will be extended in form of a second-order granularity adjustment in order to account for small sized portfolios.[167] The possibility of such an extension was already mentioned by Gordy (2004) but neither derived nor tested

[164]Gordy (2003) comes to the conclusion that the granularity adjustment works fine for risk buckets of more than 200 loans considering low credit quality buckets and for more than 1,000 loans for high credit quality buckets. However, he uses the CreditRisk$^+$ framework from Credit Suisse Financial Products (1997) and not the Vasicek model that builds the basis of Basel II, and he does not analyze the effect of different correlation factors as they are assumed in Basel II.

[165]This question is also interesting when analyzing the Basel II formula because the designated add-on factor for the potential violation of assumption (A) was cancelled from the second consultative document to the third consultative document; see BCBS (2001a, 2003a). Thus, we only prove under which conditions the assumption (A) of the Vasicek model is fulfilled. Of course, this model may suffer from other assumptions like the distributional assumption of standardized returns. However, since we would only like to address the topic of concentration risk, our focus should be reasonable. Additionally, the distributional assumptions seem not to have a deep impact on the measured VaR; see Koyluoglu and Hickman (1998a, b), Gordy (2000) or Hamerle and Rösch (2005a, b, 2006).

[166]Wilde (2001) calls this "the granularity adjustment to first order in the unsystematic variance".

[167]This procedure can be motivated by the fact that for market risk quantification of nonlinear exposures two factors of the Taylor series (fist and second order) are common to achieve a higher accuracy; see e.g. Crouhy et al. (2001) or Jorion (2001). This might be appropriate for credit risk as well. Furthermore, the higher order derivatives of VaR given by Wilde (2003) make it possible to systematically derive such a formula.

so far. Secondly, the minimum number of loans in a portfolio will be inferred numerically using two definitions of accuracy in order to enhance the theoretical background with concrete facts on critical portfolio sizes.[168] This could give an advice which sub-portfolios have significant risk concentrations and thus should be controlled on credit portfolio and not on individual credit level. In the first analyses it will be focused on homogeneous credit portfolios, i.e. each borrower has an identical PD as well as an identical EAD and LGD. Furthermore, the granularity adjustment of an inhomogeneous portfolio will be examined on the basis of Monte Carlo simulations as well. These analyses contribute to the explanation of differences between simulated and analytically determined solutions to credit portfolio risk as well as between Basel II capital requirements for Pillar 2 with respect to Pillar 1.[169]

Although it could be shown that the non-coherency of the VaR is not relevant for the ASRF model, this result does not hold anymore if the assumption of infinite granularity is not fulfilled. Thus, in Sect. 4.3 the derivation of the granularity adjustment and the aforementioned numerical analyses will be performed for the ES as well. In addition, the performance of the ES-based granularity adjustment will be tested for portfolios with stochastic LGDs. Beside the theoretical advantages of the ES, the results of the numerical study demonstrate that the granularity adjustment generates better approximations for the ES than for the VaR. Moreover, even if stochastic LGDs are included as an additional source of uncertainty, the accuracy of the adjustment formula is very high.

4.2 Measurement of Name Concentration Using the Risk Measure Value at Risk[170]

4.2.1 Considering Name Concentration with the Granularity Adjustment

4.2.1.1 First-Order Granularity Adjustment for One-Factor Models

The principle of incorporating the effect of the portfolio size in a one-factor model is very simple. As a first step, it is assumed that the portfolio is infinitely fine

[168]The Basel Committee on Banking Supervision already stated that in principle the effect of portfolio size on credit risk is well understood but lacks practical analyses; see BCBS (2005b).

[169]Additionally, this study makes contribution to the ongoing research on analyzing differences between Basel II capital requirements and banks internal “true” risk capital measurement approaches. Since the harmonization of the regulatory capital requirements and the perceived risk capital of banks internal estimates for portfolio credit risk is often stated as the major benefit of Basel II, see e.g. Hahn (2005), p. 127, but often not observed in practice, this task might be of relevance in the future.

[170]The main results of this section comply with Gürtler et al. (2008a).

grained and the VaR can be determined under the ASRF framework. However, an add-on factor is constructed, which accounts for the finite size of the portfolio and converges to zero if assumption (A) of infinite granularity is (nearly) met. This factor can be determined in form of the first element different from zero that results from a Taylor series expansion of the VaR around the ASRF solution. An alternative approach is to evaluate the unintentional shift of the confidence level due to the negligence of granularity and to transform the result into a shift of the loss quantile. The approximation is based on some linearizations around the systematic loss. Hence, both approaches rely on the proximity of the true VaR and the VaR under the ASRF framework. As the implementation of the Taylor series expansion is more straightforward, the following explanations are referred to this approach. The pioneer work on the granularity adjustment of Wilde (2001), which relies on the other approach mentioned, is presented in Appendix 4.5.1.

In order to perform the Taylor series expansion, the portfolio loss will be subdivided into a systematic and an unsystematic part, i.e.

$$\tilde{L} = \mathbb{E}\left(\tilde{L}\,|\,\tilde{x}\right) + \left[\tilde{L} - \mathbb{E}\left(\tilde{L}\,|\,\tilde{x}\right)\right] =: \tilde{Y} + \lambda\tilde{Z}. \tag{4.1}$$

Thus, the first term $\mathbb{E}(\tilde{L}\,|\,\tilde{x}) =: \tilde{Y}$ describes the systematic part of the portfolio loss that can be expressed as the expected loss conditional on $\tilde{x}$ (see also (2.85)). The second term $\tilde{L} - \mathbb{E}(\tilde{L}\,|\,\tilde{x}) =: \lambda\tilde{Z}$ of (4.1) stands for the unsystematic part of the portfolio loss, which results from the idiosyncratic risk. Therefore, $\tilde{Z}$ describes the general idiosyncratic component and λ decides on the fraction of the idiosyncratic risk that stays in the portfolio. Obviously, λ tends to zero if the number of obligors n converges to infinity, since this fraction (of the idiosyncratic risk) vanishes if granularity assumption (A) from Sect. 2.6 holds. However, for a granularity adjustment we claim that the portfolio is only "nearly" infinitely granular and thus λ is just close to but exceeds zero. In order to incorporate the idiosyncratic part of the portfolio loss into the VaR-formula, we perform a *Taylor series expansion around the systematic loss* at $\lambda = 0$. We get

$$\begin{aligned} VaR_{\alpha}\left(\tilde{L}\right) &= VaR_{\alpha}\left(\tilde{Y} + \lambda\tilde{Z}\right) \\ &= VaR_{\alpha}\left(\tilde{Y}\right) + \lambda\left[\frac{dVaR_{\alpha}\left(\tilde{Y} + \lambda\tilde{Z}\right)}{d\lambda}\right]_{\lambda=0} + \frac{\lambda^2}{2!}\left[\frac{d^2VaR_{\alpha}\left(\tilde{Y} + \lambda\tilde{Z}\right)}{d\lambda^2}\right]_{\lambda=0} \\ &\quad + \cdots + \frac{\lambda^m}{m!}\left[\frac{d^mVaR_{\alpha}\left(\tilde{Y} + \lambda\tilde{Z}\right)}{d\lambda^m}\right]_{\lambda=0} + \cdots \ . \end{aligned} \tag{4.2}$$

Thus, the first term describes the systematic part of the VaR and all other terms add an additional fraction to the VaR due to the undiversified idiosyncratic component. If the Taylor series expansion is formed up to the quadratic term, the first two derivatives of VaR are needed. According to Gouriéroux et al. (2000), the *first and second derivative of VaR* are given as[171]

$$\left.\frac{dVaR_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}{d\lambda}\right|_{\lambda=0} = \mathbb{E}\left[\tilde{Z}|\tilde{Y}=q_\alpha\left(\tilde{Y}\right)\right], \tag{4.3}$$

$$\left.\frac{d^2VaR_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}{d^2\lambda}\right|_{\lambda=0} = -\frac{1}{f_Y(y)}\frac{d}{dy}\left(f_Y(y)\,\mathbb{V}\left[\tilde{Z}\,|\,\tilde{Y}=y\right]\right)\Big|_{y=q_\alpha\left(\tilde{Y}\right)}, \tag{4.4}$$

with $f_Y(y)$ being the probability density function of $\tilde{Y}$. Concurrently, the first derivative of VaR equals zero[172]:

$$\mathbb{E}\left(\tilde{Z}\,|\,\tilde{Y}\right) = \frac{1}{\lambda}\cdot\mathbb{E}\left(\tilde{L}-\mathbb{E}\left(\tilde{L}\,|\,\tilde{x}\right)\,|\,\tilde{Y}\right) = \frac{1}{\lambda}\cdot\mathbb{E}\left(\tilde{L}\,|\,\tilde{Y}\right) - \frac{1}{\lambda}\cdot\mathbb{E}\left(\tilde{L}\,|\,\tilde{Y}\right) = 0, \tag{4.5}$$

so that the second derivative is the first relevant element underlying the granularity adjustment. With

$$\lambda^2\cdot\mathbb{V}\left[\tilde{Z}\,|\,\tilde{Y}\right] = \mathbb{V}\left[\lambda\tilde{Z}\,|\,\tilde{Y}\right] = \mathbb{V}\left[\tilde{L}-\tilde{Y}\,|\,\tilde{Y}\right] = \mathbb{V}\left[\tilde{L}\,|\,\tilde{Y}\right], \tag{4.6}$$

the quadratic term of the Taylor series expansion (4.2) results in

$$\begin{aligned}\Delta l_1 &= \frac{\lambda^2}{2}\left(-\frac{1}{f_Y(y)}\frac{d}{dy}\left(f_Y(y)\,\mathbb{V}\left[\tilde{Z}\,|\,\tilde{Y}=y\right]\right)\Big|_{y=q_\alpha\left(\tilde{Y}\right)}\right)\\ &= -\frac{1}{f_Y(y)}\frac{d}{dy}\left(f_Y(y)\,\mathbb{V}\left[\tilde{L}\,|\,\tilde{Y}=y\right]\right)\Big|_{y=q_\alpha\left(\tilde{Y}\right)}.\end{aligned} \tag{4.7}$$

As the conditional expectation $\tilde{Y}=\mathbb{E}(\tilde{L}\,|\,\tilde{x})$ is continuous and strictly monotonously decreasing in x, the probability density function $f_Y(y)$ can be transformed into[173]

$$f_Y(y) = \frac{f_x(x)}{|dy/dx|} = -\frac{f_x(x)}{dy/dx} = -\frac{f_x(x)}{\frac{d}{dx}\mathbb{E}\left(\tilde{L}\,|\,\tilde{x}=x\right)}. \tag{4.8}$$

[171] See Appendix 4.5.2.

[172] This is valid because the added risk of the portfolio is unsystematic; see Martin and Wilde (2002) for further explanations.

[173] See Appendix 4.5.3.

Furthermore, using (4.8) and[174]

$$\begin{aligned}
&\tilde{Y} = q_\alpha\left(\tilde{Y}\right) \\
&\quad \Leftrightarrow \mathbb{E}\left(\tilde{L} \,|\, \tilde{x}\right) = q_\alpha\left(\mathbb{E}\left(\tilde{L} \,|\, \tilde{x}\right)\right) \\
&\quad \Leftrightarrow \mathbb{E}\left(\tilde{L} \,|\, \tilde{x}\right) = \mathbb{E}\left(\tilde{L} \,|\, q_{1-\alpha}(\tilde{x})\right) \\
&\quad \Leftrightarrow \tilde{x} = q_{1-\alpha}(\tilde{x}),
\end{aligned} \tag{4.9}$$

the true quantile of a granular portfolio $VaR_\alpha^{(n)}$ can be approximated by the Taylor series expansion up to the quadratic term, which leads to the following formula for the VaR including the *granularity adjustment* Δl_1:

$$\begin{aligned}
&VaR_\alpha^{(n)} \approx VaR_\alpha^{(\text{ASRF})} + \Delta l_1 =: VaR_\alpha^{(\text{1st Order Adj.})} \\
&\text{with } \Delta l_1 = -\frac{1}{2f_x(x)} \frac{d}{dx}\left(\frac{f_x(x)\mathbb{V}\left[\tilde{L} \,|\, \tilde{x} = x\right]}{\frac{d}{dx}\mathbb{E}\left[\tilde{L} \,|\, \tilde{x} = x\right]} \right)\Bigg|_{x = q_{1-\alpha}(\tilde{x})}.
\end{aligned} \tag{4.10}$$

This corresponds to the result of Wilde (2001) and Rau-Bredow (2002). Thus, the VaR figure of the infinitely fine grained portfolio is adjusted by an additional term, that is the first term different from zero of the Taylor series expansion (4.2). In contrast to the ASRF solution, which relies on the conditional expectation only, the granularity adjustment takes the conditional variance of the portfolio loss into account. In the following, the expression above will be called the ASRF solution with first-order (granularity) adjustment.

A more detailed analysis of (4.10) will show that the granularity adjustment is a term of order $O(1/n^*)$, or for homogeneous portfolios simply $O(1/n)$.[175] For this purpose, the conditional expectation and variance will be looked at. Due to the conditional independence of the credit events and due to the restriction of the individual loss rate $(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}})$ to $[-1,\ 1]$ for all $i \in \{1,\ \ldots,\ n\}$, there exists a finite number $V^*(x) \leq 1$ such that

$$\begin{aligned}
\mathbb{V}\left(\tilde{L} \,|\, \tilde{x} = x\right) &= \mathbb{V}\left(\sum_{i=1}^{n} w_i \cdot \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} | \tilde{x} = x\right) = \sum_{i=1}^{n} {w_i}^2 \cdot \mathbb{V}\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} | \tilde{x} = x\right) \\
&= \sum_{i=1}^{n} {w_i}^2 \cdot V^*(x) = V^*(x) \cdot \sum_{i=1}^{n} {w_i}^2 = V^*(x) \cdot \frac{1}{n^*}.
\end{aligned} \tag{4.11}$$

[174] Cf. the identity 2.90.

[175] The notation n^* refers to the effective number of credits as introduced in (2.87).

Under the same conditions there also exists a finite number $E^*(x) \leq 1$ such that

$$\begin{aligned}\mathbb{E}\left(\tilde{L} \,|\, \tilde{x} = x\right) &= \mathbb{E}\left(\sum_{i=1}^{n} w_i \cdot \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} | \tilde{x} = x\right) = \sum_{i=1}^{n} w_i \cdot \mathbb{E}\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} | \tilde{x} = x\right) \\ &= E^*(x) \cdot \sum_{i=1}^{n} w_i = E^*(x). \end{aligned} \tag{4.12}$$

Using these expressions, the granularity add-on Δl_1 from (4.10) can be written as

$$\Delta l_1 = -\frac{1}{n^*} \frac{1}{2f_x(x)} \frac{d}{dx} \left(\frac{f_x(x) V^*(x)}{\frac{d}{dx} E^*(x)} \right) \bigg|_{x = q_{1-\alpha}(\tilde{x})} = O\left(\frac{1}{n^*}\right). \tag{4.13}$$

This shows that the granularity adjustment is linear in terms of $1/n^*$, so that in a homogeneous portfolio the add-on for undiversified idiosyncratic risk is halved if the number of credits is doubled. This corresponds to the heuristic approach of Gordy (2001), who presumed that the add-on is constant in terms of $1/n$ and estimated the slope of this term by simulation. At the same time it has to be stated that neglecting the additional terms of the Taylor series expansion, which are at least of order $O(1/n^2)$ in the homogeneous case,[176] implies that all higher moments like the conditional skewness and kurtosis are ignored. This can be made clear by expressing the higher conditional moments about the mean η_m similar to (4.11) and (4.12) as[177]

$$\begin{aligned}\eta_m\left(\tilde{L} \,|\, \tilde{x} = x\right) &= \sum_{i=1}^{n} {w_i}^m \cdot \eta_m\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} | \tilde{x} = x\right) = {\eta_m}^*(x) \cdot \sum_{i=1}^{n} {w_i}^m \\ &\leq {\eta_m}^*(x) \cdot \sum_{i=1}^{n} \left(\frac{b}{n \cdot a}\right)^m = {\eta_m}^*(x) \cdot \left(\frac{b}{a}\right)^m \cdot \frac{1}{n^{m-1}} \\ &= O\left(\frac{1}{n^{m-1}}\right), \end{aligned} \tag{4.14}$$

with some finite numbers ${\eta_m}^*(x) \leq 1$ and $0 < a \leq EAD_i \leq b$ for all i. If higher moments like the conditional skewness shall be considered for the granularity adjustment, too, it would be necessary to include additional elements of the Taylor series expansion. This will be done in the subsequent Sect. 4.2.1.3, but beforehand, the first-order granularity adjustment will be applied to the Vasicek model.

[176]The equivalent term for heterogeneous portfolios is $O\left(\sum_{i=1}^{n} w^3\right)$.

[177]The mth moment of a random variable $\tilde{X}$ about the mean $\eta_m(\tilde{X})$ is defined as $\eta_m(\tilde{X}) := \mathbb{E}([\tilde{X} - \mathbb{E}(\tilde{X})]^m)$; cf. Abramowitz and Stegun (1972), 26.1.6.

4.2.1.2 First-Order Granularity Adjustment for the Vasicek Model

Formula (4.10) is the general result of the granularity adjustment for one-factor models, which could be applied to different models. The application to the one-factor version of CreditRisk$^+$ is demonstrated in Wilde (2001). In the following, the granularity add-on will be specified for the Vasicek model. Thus, the conditional probability of default is assumed to be given by

$$p_i(x) = \Phi\left(\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i} \cdot x}{\sqrt{1-\rho_i}}\right) \tag{4.15}$$

and the systematic factor $f_x(x) = \varphi$ is standard normally distributed. For ease of notation, the mth moment of some random variable $\tilde{X}$ about the origin will be denoted by $\mu_m(\tilde{X}) := \mathbb{E}(\tilde{X}^m)$, and the mth conditional moment of the portfolio loss about the origin will be indicated by

$$\mu_{m,c} := \mu_m\left(\tilde{L}\,|\,\tilde{x} = x\right). \tag{4.16}$$

As noticed before, the mth moment of a random variable $\tilde{X}$ about the mean is represented by $\eta_m(\tilde{X}) := \mathbb{E}([\tilde{X} - \mathbb{E}(\tilde{X})]^m)$, and the mth conditional moment of the portfolio loss about the mean will be denoted by

$$\eta_{m,c} := \eta_m\left(\tilde{L}\,|\,\tilde{x} = x\right). \tag{4.17}$$

Using this notation, the conditional expectation and the conditional variance are indicated by $\mu_{1,c}$ and $\eta_{2,c}$, respectively, and the granularity adjustment (4.10) can be expressed as[178]

$$\begin{aligned} \Delta l_1 &= -\frac{1}{2\varphi}\frac{d}{dx}\left(\frac{\varphi\,\eta_{2,c}}{d\mu_{1,c}/dx}\right)\Bigg|_{x=\Phi^{-1}(1-\alpha)} \\ &= \frac{1}{2}\left[\frac{x \cdot \eta_{2,c}}{d\mu_{1,c}/dx} - \frac{d\eta_{2,c}/dx}{d\mu_{1,c}/dx} + \frac{\eta_{2,c} \cdot d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right]\Bigg|_{x=\Phi^{-1}(1-\alpha)}. \end{aligned} \tag{4.18}$$

Thus, the first and second derivatives of the conditional expectation as well as the first derivative of the conditional variance have to be determined. For this purpose, it will be assumed that the LGDs are stochastically independent of each

[178]Cf. Appendix 4.5.4.

other.[179] Furthermore, the expectation and variance of LGD will be denoted by *ELGD* and *VLGD*, respectively. The required moments are given as[180]

$$\mu_{1,c} = \sum_{i=1}^{n} w_i \cdot ELGD_i \cdot p_i(x), \tag{4.19}$$

$$\eta_{2,c} = \sum_{i=1}^{n} w_i^2 \cdot \left[\left(ELGD_i^2 + VLGD_i \right) \cdot p_i(x) - ELGD_i^2 \cdot p_i^2(x) \right]. \tag{4.20}$$

Thus, the needed derivatives are given as

$$\frac{d\mu_{1,c}}{dx} = \sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \frac{d(p_i(x))}{dx}, \tag{4.21}$$

$$\frac{d^2\mu_{1,c}}{dx^2} = \sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \frac{d^2(p_i(x))}{dx^2}, \tag{4.22}$$

$$\frac{d\eta_{2,c}}{dx} = \sum_{i=1}^{n} w_i^2 \cdot \left[\left(ELGD_i^2 + VLGD_i \right) \cdot \frac{d(p_i(x))}{dx} - ELGD_i^2 \cdot \frac{d\left(p_i^2(x)\right)}{dx} \right]. \tag{4.23}$$

According to this, the first two derivatives of $p_i(x)$ as well as the first derivative of ${p_i}^2(x)$ have to be determined. Using the notation

$$p_i(x) = \Phi(z_i), \quad \text{with } z_i = \frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i}x}{\sqrt{1-\rho_i}}, \tag{4.24}$$

we obtain

$$\frac{d(p_i(x))}{dx} = \frac{d}{dx}\Phi(z_i) = -\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}} \cdot \varphi(z_i), \tag{4.25}$$

$$\frac{d^2(p_i(x))}{dx^2} = -\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}} \cdot \frac{d}{dx}\varphi(z_i) = -\frac{\rho_i}{1-\rho_i} \cdot z_i \cdot \varphi(z_i), \tag{4.26}$$

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[179]This assumption can be critical for real-world portfolios. Especially, it is often assumed in ongoing research on credit portfolio modeling that the LGD is dependent on the systematic factor. However, the granularity adjustment formula would complicate significantly as neither the ELGD nor the VLGD could be treated as constant for the derivatives. Against this background, this assumption will be retained for the derivation.

[180]Cf. Appendix 4.5.4. Pykhtin and Dev (2002) corrected the formulas of Wilde (2001), who neglected the last term of the following conditional variance.

$$\frac{d\left(p_i^2(x)\right)}{dx} = \frac{d}{dx}\left(\Phi(z_i)\right)^2 = -2 \cdot \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}} \cdot \Phi_i(z_i) \cdot \varphi(z_i). \tag{4.27}$$

Formulas (4.21)–(4.27) have to be inserted into (4.18) to get the granularity adjustment. This leads to the following expression for the first-order granularity adjustment for heterogeneous portfolios in the Vasicek model:

$$\begin{aligned}\Delta l_1 = \frac{1}{2}\Bigg[& \Phi^{-1}(\alpha)\frac{\sum_{i=1}^{n} w_i^2\left[\left(ELGD_i^2 + VLGD_i\right)\Phi(z_i) - ELGD_i^2\Phi^2(z_i)\right]}{\sum_{i=1}^{n} w_i ELGD_i \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}} \cdot \varphi(z_i)} \\ & - \frac{\sum_{i=1}^{n} w_i^2\left[\left(ELGD_i^2 + VLGD_i\right)\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\varphi(z_i) - 2ELGD_i^2\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\Phi_i(z_i)\varphi(z_i)\right]}{\sum_{i=1}^{n} w_i ELGD_i \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\varphi(z_i)} \\ & - \sum_{i=1}^{n} w_i^2\left[\left(ELGD_i^2 + VLGD_i\right)\Phi(z_i) - ELGD_i^2\Phi^2(z_i)\right] \\ & \cdot \frac{\sum_{i=1}^{n} w_i ELGD_i \frac{\rho_i}{1-\rho_i} z_i\varphi(z_i)}{\left(\sum_{i=1}^{n} w_i ELGD_i \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\varphi(z_i)\right)^2}\Bigg]_{z_i=\frac{\Phi^{-1}(PD_i)+\sqrt{\rho_i}\Phi^{-1}(\alpha)}{\sqrt{1-\rho_i}}} .\end{aligned} \tag{4.28}$$

For homogeneous portfolios, this formula can be simplified to[181]

$$\begin{aligned}\Delta l_1 = \frac{1}{2n}\Bigg(&\frac{ELGD^2 + VLGD}{ELGD}\left[\frac{\Phi(z)}{\varphi(z)}\frac{\Phi^{-1}(\alpha)(1-2\rho) + \Phi^{-1}(PD)\sqrt{\rho}}{\sqrt{\rho}\sqrt{1-\rho}} - 1\right] \\ & - ELGD \cdot \Phi(z)\left[(z)\frac{\Phi^{-1}(\alpha)(1-2\rho) + \Phi^{-1}(PD)\sqrt{\rho}}{\sqrt{\rho}\sqrt{1-\rho}} - 2\right]\Bigg)_{z=\frac{\Phi^{-1}(PD)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}},\end{aligned} \tag{4.29}$$

which is the formula presented by Pykhtin and Dev (2002).

4.2.1.3 Second-Order Granularity Adjustment for One-Factor Models

Recalling the discussion of the first-order granularity adjustment, the ASRF solution might only lead to good approximations if term (4.28) of order $O(1/n)$ is close

[181] Cf. Appendix 4.5.5.

to zero, whereas the ASRF solution including the first-order granularity adjustment might only be sufficient if the terms of order $O(1/n^2)$ vanish. For medium sized risk buckets this might be true, but if the number of credits in the portfolio is getting considerably small, an additional factor might be appropriate. Particularly, the mentioned granularity adjustment is linear in $1/n$ and this might not hold for small portfolios. Indeed, Gordy (2003) shows by simulation that the portfolio loss seems to follow a concave function and therefore adjustment (4.28) would slightly overshoot the theoretically optimal add-on for smaller portfolios.[182] An explanation of the described behavior is that the first-order adjustment only takes the conditional variance into account whereas higher conditional moments, which result from higher order terms, are ignored. As noticed in Sect. 4.1, additional elements of the Taylor series expansion (4.2) will be calculated in the following with the intention to improve the adjustment for small portfolio sizes. Hence, all elements of order $O(1/n^2)$ will be taken into account, and thus the error will be reduced to $O(1/n^3)$.[183] This newly derived formula will be called the *second-order granularity adjustment*. The resulting ASRF solution including the first and the second-order granularity adjustment Δl_2 is

$$VaR_\alpha^{(\text{1st + 2nd Order Adj.})} = VaR_\alpha^{(\text{ASRF})} + \Delta l_1 + \Delta l_2, \tag{4.30}$$

where Δl_2 represents the $O(1/n^2)$ elements of (4.2).

In order to calculate these elements, higher derivatives of VaR are required. Referring to Wilde (2003), a formula for all derivatives of VaR is derived in Appendix 4.5.6. Having a closer look at the derivatives of VaR, the fourth and a part of the fifth element of the Taylor series are identified to be relevant for the $O(1/n^2)$ terms.[184] Thus, the third and the fourth derivative of VaR are required. As shown in Appendix 4.5.7, the rather complex result for all derivatives can be simplified for the first five derivatives ($m = 1, 2, \ldots, 5$) of VaR to

$$\begin{aligned} \left.\frac{\partial^m VaR_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}{\partial\lambda^m}\right|_{\lambda=0} &= (-1)^m\left(-\frac{1}{f_Y(y)}\right)\left[\frac{d^{m-1}\left(\mu_m\left(\tilde{Z}\,|\,\tilde{Y}=y\right)f_Y(y)\right)}{dy^{m-1}}\right. \\ &\left.\quad -\kappa(m)\cdot\frac{d}{dy}\left(\frac{1}{f_Y(y)}\cdot\frac{d\left(\mu_2\left(\tilde{Z}\,|\,\tilde{Y}=y\right)f_Y(y)\right)}{dy}\frac{d^{m-3}\left(\mu_{m-2}\left(\tilde{Z}\,|\,\tilde{Y}=y\right)f_Y(y)\right)}{dy^{m-3}}\right)\right]_{y=q_\alpha\left(\tilde{Y}\right)}, \end{aligned} \tag{4.31}$$

with $\kappa(1) = \kappa(2) = 0,\ \kappa(3) = 1,\ \kappa(4) = 3$, and $\kappa(5) = 10$.

[182] Gordy (2003) observes the concavity of the granularity add-on for a high-quality portfolio (A-rated) up to a portfolio size of 1,000 debtors.

[183] See Gordy (2004), p. 112, footnote 5, for a similar suggestion.

[184] See Appendix 4.5.8 for details regarding the order of these elements.

Using the third and the fourth derivative of VaR and due to[185]

$$\lambda^m \cdot \mu_m\left(\tilde{Z}\,|\,\tilde{Y}=y\right)\big|_{y=q_\alpha\left(\tilde{Y}\right)} = \eta_m\left[\tilde{L}\,|\,\tilde{Y}=y\right]\big|_{y=q_\alpha\left(\tilde{Y}\right)} =: \eta_m(y)\big|_{y=q_\alpha\left(\tilde{Y}\right)} \tag{4.32}$$

as well as $\eta_1(y)=0$, the elements of order $O(1/n^2)$ of the Taylor series expansion (4.2) are given as

$$\begin{aligned}\Delta l_2 &= \frac{(-1)^3}{3!}\left(-\frac{1}{f_Y(y)}\right)\left[\frac{d^2(\eta_3(y)f_Y(y))}{dy^2} - \frac{d}{dy}\left(\frac{1}{f_Y(y)}\frac{d(\eta_2(y)f_Y(y))}{dy}(\eta_1(y)f_Y(y))\right)\right] \\ &\quad + \frac{(-1)^4}{4!}\left(-\frac{1}{f_Y(y)}\right)\left[-3\frac{d}{dy}\left(\frac{1}{f_Y(y)}\frac{d(\eta_2(y)f_Y(y))}{dy}\frac{d(\eta_2(y)f_Y(y))}{dy}\right)\right]\Bigg|_{y=q_\alpha\left(\tilde{Y}\right)} \\ &= \frac{1}{6}\frac{1}{f_Y(y)}\frac{d^2}{dy^2}[\eta_3(y)f_Y(y)] + \frac{1}{24}\frac{3}{f_Y(y)}\frac{d}{dy}\left[\frac{1}{f_Y(y)}\left(\frac{d}{dy}[\eta_2(y)f_Y(y)]\right)^2\right]\Bigg|_{y=q_\alpha\left(\tilde{Y}\right)}.\end{aligned} \tag{4.33}$$

Recalling that $\mu_{m,c}=\mu_m(\tilde{L}\,|\,\tilde{x}=x)$, $f_Y(y)=-\frac{f_x(x)}{dy/dx}$ (see (4.8)), and $\eta_m(y)|_{y=q_\alpha(\tilde{Y})} := \eta_m(\tilde{L}\,|\,\tilde{Y}=q_\alpha(\tilde{Y})) = \eta_m(\tilde{L}\,|\,\tilde{x}=q_{1-\alpha}(\tilde{x})) =: \eta_{m,c}\big|_{x=q_{1-\alpha}(\tilde{x})}$ (cf. (4.9) and (4.32)), Δl_2 can be written as

$$\begin{aligned}\Delta l_2 &= \frac{1}{6f_x}\frac{d}{dx}\left(\frac{d}{dy}\left[\frac{\eta_{3,c}f_x}{dy/dx}\right]\right) + \frac{1}{8f_x}\frac{d}{dx}\left[\frac{1}{f_x}\frac{dy}{dx}\left(\frac{d}{dy}\left[\frac{\eta_{2,c}f_x}{dy/dx}\right]\right)^2\right]\Bigg|_{x=q_{1-\alpha}(\tilde{x})} \\ &= \frac{1}{6f_x}\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\frac{d}{dx}\left[\frac{\eta_{3,c}f_x}{d\mu_{1,c}/dx}\right]\right) \\ &\quad + \frac{1}{8f_x}\frac{d}{dx}\left[\frac{1}{f_x}\frac{1}{d\mu_{1,c}/dx}\left(\frac{d}{dx}\left[\frac{\eta_{2,c}f_x}{d\mu_{1,c}/dx}\right]\right)^2\right]\Bigg|_{x=q_{1-\alpha}(\tilde{x})},\end{aligned} \tag{4.34}$$

which is our general result for the second-order granularity adjustment. Having a closer look at (4.34), it can be seen that the second-order adjustment takes a squared term of the conditional variance as well as the conditional skewness into account,[186] which are both of order $O(1/n^2)$.[187]

[185]Cf. (4.236) of Appendix 4.5.8.

[186]Precisely, the element $\eta_{3,c}$ is the third conditional moment centered about the mean whereas the conditional skewness is the "normalized" third moment, defined as the third conditional moment about the mean divided by the conditional standard deviation to the power of three.

[187]Cf. (4.14).

4.2.1.4 Second-Order Granularity Adjustment for the Vasicek Model

Similar to Sect. 4.2.1.2, we specify our general result of the second-order granularity adjustment for the Vasicek model with

$$p_i(x) = \Phi\left(\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i}\cdot x}{\sqrt{1-\rho_i}}\right) \tag{4.35}$$

and a standard normally distributed systematic factor, leading to $f_x = \varphi$ and $q_{1-\alpha}(\tilde{x}) = \Phi^{-1}(1-\alpha)$. As derived in Appendix 4.5.9 under the assumption of a standard normally distributed systematic factor, the second-order granularity adjustment is equivalent to

$$\begin{aligned}
\Delta l_2 = & \frac{1}{6\left(d\mu_{1,c}/dx\right)^2}\left[\eta_{3,c}\left(x^2-1-\frac{d^3\mu_{1,c}/dx^3}{d\mu_{1,c}/dx}+\frac{3x\left(d^2\mu_{1,c}/dx^2\right)}{d\mu_{1,c}/dx}+\frac{3\left(d^2\mu_{1,c}/dx^2\right)^2}{\left(d\mu_{1,c}/dx\right)^2}\right)\right.\\
& \left.+\frac{d\eta_{3,c}}{dx}\left(-2x-\frac{3\left(d^2\mu_{1,c}/dx^2\right)}{d\mu_{1,c}/dx}\right)+\frac{d^2\eta_{3,c}}{dx^2}\right]\\
& +\frac{1}{8\left(d\mu_{1,c}/dx\right)^3}\left[\left(-x-3\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\left(\eta_{2,c}\left[-x-\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]+\frac{d\eta_{2,c}}{dx}\right)^2\right.\\
& +2\left(\eta_{2,c}\left[x+\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]-\frac{d\eta_{2,c}}{dx}\right)\left(\eta_{2,c}\left[1+\frac{d^3\mu_{1,c}/dx^3}{d\mu_{1,c}/dx}-\frac{\left(d^2\mu_{1,c}/dx^2\right)^2}{\left(d\mu_{1,c}/dx\right)^2}\right]\right.\\
& \left.\left.\left.+\frac{d\eta_{2,c}}{dx}\left[x+\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]-\frac{d^2\eta_{2,c}}{dx^2}\right)\right]\right|_{x=\Phi^{-1}(1-\alpha)}.
\end{aligned} \tag{4.36}$$

As can be seen from (4.36), Δl_2 is a function of $\mu_{1,c}$, $\eta_{2,c}$, and $\eta_{3,c}$. According to (4.19), (4.20), and (4.264),[188] these moments are given as

$$\mu_{1,c} = \sum_{i=1}^{n} w_i \cdot ELGD_i \cdot p_i(x), \tag{4.37}$$

$$\eta_{2,c} = \sum_{i=1}^{n} w_i^2 \cdot \left[\left(ELGD_i^2 + VLGD_i\right)\cdot p_i(x) - ELGD_i^2 \cdot p_i^2(x)\right], \tag{4.38}$$

[188] See Appendix 4.5.10.

$$\eta_{3,c} = \sum_{i=1}^{n} w_i^3 \big[\big(ELGD_i^3 + 3 \cdot ELGD_i \cdot VLGD_i + SLGD_i \big) \cdot p_i(x) \\ -3 \cdot \big(ELGD_i^3 + ELGD_i \cdot VLGD_i \big) \cdot p_i^2(x) + 2 \cdot ELGD_i^3 \cdot p_i^3(x) \big], \quad (4.39)$$

with $SLGD := \eta_3(\widetilde{LGD})$. The conditional PD from (4.35) can be written as

$$p_i(x) = \Phi(z_i), \quad \text{with } z_i = \frac{\Phi^{-1}(PD_i)}{\sqrt{1-\rho_i}} - s_i \cdot x \quad \text{and} \quad s_i = \frac{\sqrt{\rho}_i}{\sqrt{1-\rho_i}}. \quad (4.40)$$

Using this notation and having a closer look at (4.36) and the conditional moments, we find that the following derivatives are needed

$$\frac{d(p_i(x))}{dx} = -s_i \cdot \varphi(z_i), \quad (4.41)$$

$$\frac{d^2(p_i(x))}{dx^2} = -s_i^2 \cdot z_i \cdot \varphi(z_i), \quad (4.42)$$

$$\frac{d^3(p_i(x))}{dx^3} = -s_i^3 \cdot \varphi(z_i) \cdot \big(z_i^2 - 1\big), \quad (4.43)$$

$$\frac{d\big(p_i^2(x)\big)}{dx} = -2 \cdot s_i \cdot \Phi(z_i) \cdot \varphi(z_i), \quad (4.44)$$

$$\frac{d^2\big(p_i^2(x)\big)}{dx^2} = 2 \cdot s_i^2 \cdot \varphi(z_i) \cdot [\varphi(z_i) - \Phi(z_i) \cdot z_i], \quad (4.45)$$

$$\frac{d\big(p_i^3(x)\big)}{dx} = -3 \cdot s_i \cdot \Phi^2(z_i) \cdot \varphi(z_i), \quad (4.46)$$

$$\frac{d^2\big(p_i^3(x)\big)}{dx^2} = 3 \cdot s_i^2 \cdot \Phi(z_i) \cdot \varphi(z_i) \cdot [2 \cdot \varphi(z_i) - \Phi(z_i) \cdot z_i]. \quad (4.47)$$

Finally, we just have to use (4.37)–(4.47) in order to determine the second-order adjustment formula (4.36). The resulting expression can easily be calculated with standard computer applications without the need to aggregate the terms to a single formula. Thus, we have achived our aim to derive a formula that takes the conditional skewness into account and reduces the error to $O(\sum_{i=1}^{n} w^4)$ or to $O(1/n^3)$ for homogeneous portfolios. This can best be seen for homogeneous portfolios for the special case that the gross loss rates are modeled:

$$\Delta l_2 = \frac{1}{6n^2 s^2 \varphi^2} \big[\big(x^2 - 1 + s^2 + 3xsz + 2s^2 z^2\big)\big(\Phi - 3\Phi^2 + 2\Phi^3\big) \\ + s\varphi(2x + 3sz)\big(1 - 6\Phi + 6\Phi^2\big) - s^2\varphi(z - 6[\Phi z - \varphi] + 6\Phi[\Phi z - 2\varphi])\big]$$

$$
\begin{aligned}
&-\frac{1}{8n^2s^3\varphi^3}\Big[(-x-3sz)\big(\big[\Phi-\Phi^2\big][-x-sz]-s\varphi[1-2\Phi]\big)^2 \\
&+2\big(\big[\Phi-\Phi^2\big][x+sz]+s\varphi[1-2\Phi]\big) \\
&\cdot\big(\big[\Phi-\Phi^2\big]\big[1-s^2\big]-s\varphi[1-2\Phi][x+sz]+s^2\varphi[z+2(\varphi-\Phi z)]\big)\Big], \qquad (4.48)
\end{aligned}
$$

with $\Phi=\Phi(z)$, $\varphi=\varphi(z)$, $z=\frac{\Phi^{-1}(PD)-\sqrt{\rho}\cdot x}{\sqrt{1-\rho}}$, $s=\frac{\sqrt{\rho}}{\sqrt{1-\rho}}$, and $x=\Phi^{-1}(1-\alpha)$.

Even if the formulas appear quite complex, both adjustments are easy to implement, fast to compute and we do not have to run Monte Carlo simulations and thereby avoid simulation noise.

4.2.2 Numerical Analysis of the VaR-Based Granularity Adjustment

4.2.2.1 Impact on the Portfolio-Quantile

As mentioned in Sect. 4.1, there is no concrete analysis in the literature for which type of credit portfolios the impact of portfolio name concentrations is negligible. Instead, we only essentially know that a (homogeneous) portfolio consisting of a higher number of credits incorporates less name concentration risk or that name concentrations can account for round about 13–21% additional risk if the portfolio is highly concentrated.[189] Moreover, we do not know how good the first-order or the second-order granularity adjustment formulas work for different portfolio types. Against this background, subsequently the accuracy of the ASRF formula, the first-order, and the second-order granularity adjustment will be analyzed.

At first, we discuss the general behavior of the four procedures for risk quantification of homogeneous portfolios presented in Sects. 2.5, 2.6, 2.7, 4.2.1.2, and 4.2.1.4, which are

(a) The numerically “exact” coarse grained solution (see (2.75))
(b) The fine grained ASRF solution (see (2.97))
(c) The ASRF solution with first-order adjustment (see (4.10) and (4.29))
(d) The ASRF solution with first- and second-order adjustments (see (4.30) and (4.48))

each applying the conditional probability of default (2.66) of the Vasicek model. For the comparison, we evaluate the portfolio loss distribution of a simple portfolio

[189] Cf. BCBS (2006), p. 10.

that consists of 40 credits, each with a probability of default of $PD = 1\%$ and a loss given default of $LGD = 1$. The correlation parameter is set to $\rho = 20\%$.[190] Using these parameters, we calculate the loss distribution using the "exact" solution (a) as well as the approximations (b) to (d). The results are shown in Fig. 4.1 for portfolio losses up to 30 % (12 credits) and the corresponding quantiles (of the loss distribution) starting at $\alpha = 0.7$. See Fig. 4.2 for the region of high quantiles $\alpha \geq 0.994$, which are of special interest in a VaR-framework for credit risk with high confidence levels.

It is obvious to see that the coarse grained solution (a) is not continuous since the distribution of defaults is a discrete binomial mixture whereas all other solutions (b) to (d) are "smooth" functions. This is caused by the fact that these approximations for the loss distribution assume an infinitely granular portfolio, i.e. the loss distribution is monotonous increasing and differentiable (solution (b)), or at least are derived from such an idealized portfolio ((c) and (d)).

Now, we examine the result for the VaR-figures at confidence levels 0.995 and 0.999. Using the exact, discrete solution (a), the VaR is 12.5% (or 5 credits) for the

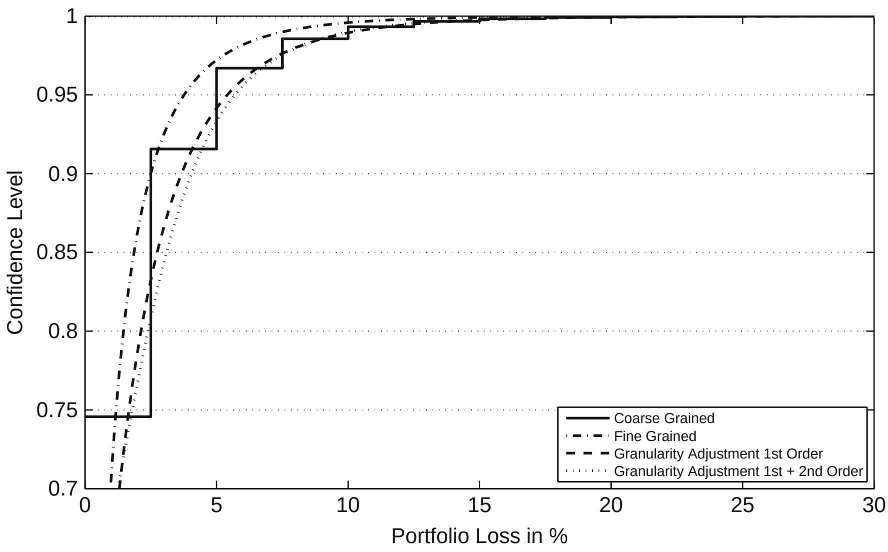

Fig. 4.1 Value at Risk for a wide range of probabilities

[190] The chosen portfolio exhibits high unsystematic risk and therefore serves as a good example in order to explain the differences of the four solutions. However, we evaluated several portfolios and basically, the results do not differ. Additionally, we claim that the general statements can also be applied to heterogeneous portfolios.

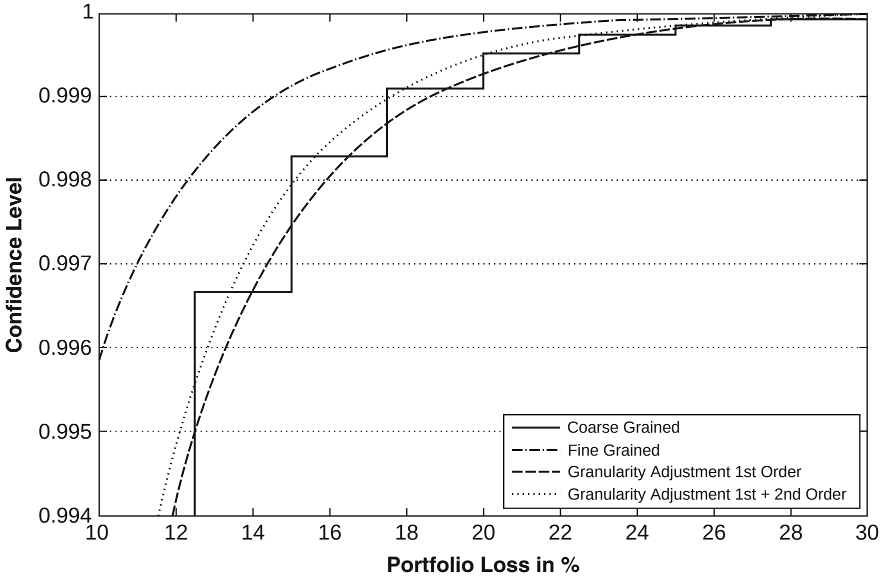

Fig. 4.2 Value at Risk for high confidence levels

0.995 quantile and 17.5% (or 7 credits) for the 0.999 quantile. Compared to this, the ASRF solution (b) exhibits significant lower losses at these confidence levels, which are 9.46% for the 0.995 quantile and 14.55% for the 0.999 quantile. Obviously, the ASRF solution underestimates the portfolio loss, since it does not take (additional) concentration risks into account. If we add the first order adjustment (c), the VaR figures increase compared to the ASRF solution (b) with values 12.55% for the 0.995 quantile and 18.59% for the 0.999 quantile. Both values are good proxies for the "true" solution (a). Especially the VaR at 0.995 confidence level is nearly exact (12.55% compared to 12.5%). However, (c) seems to be a conservative measure, since the VaR is positively biased.

Using the additional second-order adjustment (d), the VaR is lowered to 12.12% for the 0.995 quantile and 17.48% for the 0.999 quantile. In this case, the VaR at 0.999 confidence level is nearly exact (17.48% compared to 17.5%). Nonetheless, (d) is likely to be a progressive approximation for the "exact" solution (a), since the VaR is negatively biased. Summing up these first results (see also Figs. 4.1 and 4.2), using the ASRF solution (b), the portfolio distributions shift to lower losses for the VaR compared to the "exact" solution (a), since an infinitely high number of credits is presumed. Precisely, the idiosyncratic risk is diversified completely, resulting in a lower portfolio loss at high confidence levels. If the first order granularity adjustment (c) is incorporated, this effect is weakened and especially for the relevant high confidence levels the portfolio loss increases compared to the ASRF solution (b). This means that the first-order granularity adjustment is usually positive.[191] However, if the second-order granularity adjustment (d) is added, the portfolio loss distribution shifts backwards again (for high confidence levels). This can be addressed to the alternating sign of the Taylor series, as can be seen in (4.31). Since the first-order granularity adjustment is positive, the second-order adjustment tends to be negative. Thus, with incorporation of the second-order adjustment (d), the approximation of the discrete distribution of the coarse grained portfolio (a) is (in general) less conservative compared to the (only) use of the first order adjustment. However, a clear conclusion that the application of the second-order adjustment (d) in order to approximate the discrete numerical derived distribution (a) for high confidence levels outperforms the only use of the first-order adjustment (c) cannot be stated.[192]

To conclude, if we appraise the approximations for the coarse grained portfolio, we find both adjustments (c) and (d) to be a much better fit of the numerical solution in the (VaR relevant) tail region of the loss distribution than the ASRF solution, whereas the first-order adjustment is more conservative and seems to give the better overall approximation in general.

4.2.2.2 Size of Fine Grained Risk Buckets

Reconsidering the assumptions of the ASRF framework (see Sect. 2.6), we found assumption (A) – the infinite granularity assumption – to be critical in a one factor model. Thus, we investigate in detail the critical numbers of credits in homogeneous portfolios that fulfill this condition. Therefore, we have to define a critical value for the deviation of the "idealized" VaR of the ASRF solution (b) from the "true" VaR figure from solution (a) to discriminate an infinite granular portfolio from a finite granular portfolio. We do that in two ways:

Firstly, it could be argued that the fine grained approximation (2.97) in order to calculate the VaR is only adequate if its value does not exceed the "true" VaR from (2.75) of the coarse grained bucket minus a target tolerance β, both using a confidence level of 0.999. Precisely, we define a critical number $I_{c,\text{per}}^{(\text{ASRF})}$ of credits in the bucket, so that each portfolio with a higher number of credits than $I_{c,\text{per}}^{(\text{ASRF})}$ meets this specification. We use the expression[193]

[191] See Rau-Bredow (2005) for a counter-example for very unusual parameter values. This problem can be addressed to the use of VaR as a measure of risk which does not guarantee sub-additivity; cf. Sect. 2.2.3.

[192] By contrast, we expected a significant enhancement by using the second order adjustment like mentioned in Gordy (2004), p. 112, footnote 5.

[193] To address to the minimum number after which the target tolerance will permanently hold, we have to add the notation "for all $N \geq n$" because the function of the coarse grained VaR exhibits jumps dependent on the number of credits.

$$I_{c,\text{per}}^{(\text{ASRF})} = \inf\left(n : \left| \frac{VaR_{0.999}^{(\text{ASRF})}\left(\tilde{L}\right)}{VaR_{0.999}^{(N)}\left(\tilde{L} = \frac{1}{N}\sum_{i=1}^{N} 1_{\{\tilde{D}_i\}}\right)} - 1 \right| < \beta \ \ \forall N \in \mathbb{N}^{\geq n} \right). \tag{4.49}$$

Here, we set the target tolerance β to 5%, meaning that the "true" VaR specified by coarse grained risk buckets does not differ from the analytic VaR using the fine grained solution (2.97) by more than 5% if the number of credits in the bucket reaches at least $I_{c,\text{per}}^{(\text{ASRF})}$.

Secondly, the fine grained approximation (b) of the VaR ("idealized" VaR) may be sufficient as long as its result using a confidence level of 0.999 does not exceed the "true" VaR as defined by solution (a) of the coarse grained bucket using a confidence level of 0.995, i.e.

$$I_{c,\text{abs}}^{(\text{ASRF})} = \sup\left(n : VaR_{0.999}^{(\text{ASRF})}\left(\tilde{L}\right) < VaR_{0.995}^{(n)}\left(\tilde{L}\right)\right). \tag{4.50}$$

This definition of a critical number can be justified due to the development of the IRB-capital formula in Basel II: When the granularity adjustment (of Basel II) was cancelled, simultaneously the confidence level was increased from 0.995 to 0.999.[194] Thus, the reduction of the capital requirement by neglecting granularity was roughly compensated by an increase of the target confidence level. The risk of portfolios with a high number of credits will therefore be overestimated if we assume that the actual target confidence level is 0.995, whereas the risk for a low number of credits will be underestimated. Thus, a critical number $I_{c,\text{abs}}^{(\text{ASRF})}$ of credits in the bucket exists, so that in each portfolio with a higher number of credits than $I_{c,\text{abs}}^{(\text{ASRF})}$, the VaR can be stated to be overestimated.

The critical numbers $I_{c,\text{per}}^{(\text{ASRF})}$ and $I_{c,\text{abs}}^{(\text{ASRF})}$ for homogeneous portfolios with different parameters ρ and PD are reported in Tables 4.1 and 4.2. We do not only report the critical numbers for Basel II conditions, but also a for wide range of parameter settings that might be relevant if banks internal data are used for estimating ρ. Due to the supervisory formula, this parameter is a function of PD for corporates, sovereigns, and banks as well as for Small and Medium Enterprises (SMEs) and (other) retail exposures and remains fixed for residential mortgage exposures and revolving retail exposures.[195]

With definition (4.49), the critical numbers $I_{c,\text{per}}^{(\text{ASRF})}$ vary from 23 to 35,986 credits (see Table 4.1), dependent on the probability of default PD and the correlation

[194] Beside some adjustments on the correlation parameter, these were the major changes of the IRB-formula from the second to the third consultative document; see BCBS (2001a, 2003a).

[195] See Sect. 2.7 for details. In both tables, (rounded) parameters ρ due to Basel II are marked.

Table 4.1 Critical number of credits from that ASRF solution can be stated to be sufficient for measuring the true VaR (see (4.49))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	35,986	23,985	5,389	5,184	4,105	3,176	2,057	1,390	988	478	370	205
3.5%	30,501	20,122	4,627	4,457	3,544	2,755	1,801	1,214	861	421	322	175
4.0%	26,051	17,272	4,054	3,851	3,076	2,402	1,563	1,077	760	375	295	161
4.5%	22,372	14,906	3,569	3,392	2,719	2,132	1,398	958	690	350	271	145
5.0%	19,669	13,160	3,153	3,047	2,412	1,928	1,273	866	628	320	255	128
5.5%	17,723	11,667	2,840	2,701	2,180	1,722	1,145	784	564	289	229	125
6.0%	15,715	10,590	2,611	2,442	1,977	1,566	1,032	711	515	264	205	116
6.5%	14,276	9,452	2,366	2,252	1,828	1,428	946	655	477	251	201	106
7.0%	12,730	8,637	2,148	2,045	1,665	1,327	869	615	457	226	185	101
7.5%	11,633	7,915	1,990	1,896	1,547	1,214	827	578	412	209	167	90
8.0%	10,657	7,272	1,813	1,761	1,414	1,133	762	527	389	206	160	87
8.5%	9,785	6,695	1,720	1,607	1,318	1,040	703	505	357	200	156	87
9.0%	9,222	6,176	1,571	1,498	1,231	992	660	460	338	183	143	80
9.5%	8,504	5,707	1,466	1,427	1,152	930	610	443	326	164	135	76
10.0%	7,853	5,281	1,399	1,334	1,079	873	597	419	304	157	132	68
10.5%	7,262	5,015	1,309	1,249	1,011	804	552	382	289	153	118	70
11.0%	6,900	4,655	1,226	1,170	949	756	532	376	285	144	120	65
11.5%	6,398	4,324	1,149	1,097	911	726	493	357	257	138	109	64
12.0%	6,099	4,127	1,103	1,053	838	684	466	332	254	135	107	58
12.5%	5,669	3,843	1,036	989	806	645	450	315	242	127	103	60
13.0%	5,419	3,677	974	952	759	622	435	299	226	117	94	53
13.5%	5,046	3,430	915	896	732	587	395	284	211	117	98	55
14.0%	4,701	3,290	882	843	706	555	391	288	201	110	87	52
14.5%	4,510	3,073	851	794	666	536	362	263	200	101	91	50
15.0%	4,331	2,954	822	767	629	519	344	250	195	108	84	51
15.5%	4,044	2,763	775	741	594	491	349	254	178	95	81	52
16.0%	3,892	2,661	731	717	589	476	324	226	186	100	78	44
16.5%	3,748	2,564	690	677	557	451	315	220	174	96	75	51
17.0%	3,507	2,403	668	639	540	427	299	225	159	86	67	42
17.5%	3,383	2,320	647	619	511	404	291	205	159	95	66	38
18.0%	3,167	2,241	611	585	496	403	277	200	152	80	70	33
18.5%	3,060	2,103	593	583	469	382	263	195	145	90	61	34
19.0%	2,959	2,034	576	551	456	362	250	186	142	85	65	35
19.5%	2,863	1,969	544	521	432	352	250	186	129	80	61	30
20.0%	2,685	1,850	529	507	420	343	244	173	133	77	57	31
20.5%	2,601	1,793	500	493	409	317	232	165	127	74	58	32
21.0%	2,522	1,739	487	466	377	326	227	170	131	73	51	26
21.5%	2,446	1,635	474	454	367	301	216	158	119	63	52	27
22.0%	2,297	1,587	448	442	368	302	211	163	123	64	53	28
22.5%	2,230	1,541	437	418	349	279	206	152	118	63	55	29
23.0%	2,167	1,498	413	408	350	280	191	145	113	57	53	30
23.5%	2,036	1,457	415	398	332	266	192	142	111	58	51	22
24.0%	1,980	1,371	393	388	324	252	193	132	98	54	49	23

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.); SMEs (Sales <$ 5 Mio.); Mortgage; Revolving retail; Other retail

Table 4.2 Critical number of credits from that the exact solution at confidence level 0.995 exceeds the infinite fine granularity at confidence level 0.999 (see (4.50))

	AAA up to AA− 0.03%	A− up to A+ 0.05%	BBB+ 0.32%	BBB 0.34%	BBB− 0.46%	BB+ 0.64%	BB 1.15%	BB− 1.97%	B+ 3.19%	B 8.99%	B− 13.01%	CCC up to C 30.85%
3.0%	5,499	3,885	997	1,019	786	678	464	329	255	165	143	123
3.5%	4,354	3,126	836	793	665	542	380	274	217	138	122	110
4.0%	3,428	2,508	701	666	564	428	308	227	184	118	103	94
4.5%	3,111	1,998	588	558	434	364	266	200	155	100	93	79
5.0%	2,436	1,830	490	466	404	308	230	175	138	92	83	70
5.5%	2,239	1,445	406	386	339	288	198	154	123	77	71	65
6.0%	1,724	1,338	380	361	283	244	170	135	109	74	69	57
6.5%	1,599	1,037	312	297	266	204	161	117	97	68	58	56
7.0%	1,489	968	294	280	220	193	138	112	85	62	57	50
7.5%	1,114	906	238	264	208	183	131	97	82	57	50	46
8.0%	1,044	681	225	214	197	152	111	93	72	52	46	42
8.5%	982	641	214	204	161	145	106	80	63	47	45	43
9.0%	925	605	203	194	153	119	102	77	61	46	39	41
9.5%	874	573	161	185	146	113	85	66	59	42	38	39
10.0%	621	543	154	147	140	109	82	64	51	38	37	38
10.5%	589	516	147	140	111	104	79	61	49	37	34	35
11.0%	559	368	141	134	107	100	76	52	48	36	31	30
11.5%	532	351	135	129	103	80	63	50	41	32	28	31
12.0%	507	335	130	124	99	77	61	49	40	32	30	28
12.5%	484	320	100	95	95	74	59	47	39	31	27	29
13.0%	463	306	96	92	91	72	57	46	38	28	29	26
13.5%	443	293	92	88	71	69	55	38	37	30	24	27
14.0%	425	281	89	85	68	67	44	37	31	27	26	24
14.5%	407	270	86	82	66	65	43	36	31	24	22	28
15.0%	261	260	83	79	64	50	42	35	30	21	23	21
15.5%	251	250	80	77	62	49	40	34	29	23	25	25
16.0%	242	241	77	74	60	47	39	33	24	23	21	22
16.5%	233	155	75	72	58	46	38	27	28	20	18	23
17.0%	224	149	55	70	56	44	37	26	23	22	22	19
17.5%	216	144	53	51	54	43	36	31	27	17	20	24
18.0%	209	139	51	49	53	42	28	25	22	19	18	20
18.5%	202	135	50	48	39	41	28	24	22	19	16	20
19.0%	195	130	48	46	37	40	27	24	18	16	16	21
19.5%	189	126	47	45	36	39	26	23	21	16	19	21
20.0%	183	122	46	44	35	38	26	23	21	18	17	17
20.5%	177	118	44	43	35	37	25	22	17	18	17	17
21.0%	172	115	43	41	34	27	24	22	20	14	15	18
21.5%	167	112	42	40	33	26	24	17	16	13	15	18
22.0%	162	108	41	39	32	26	23	21	16	15	13	19
22.5%	157	105	40	38	31	25	23	21	16	15	13	19
23.0%	153	102	39	37	30	24	22	16	15	15	13	14
23.5%	148	99	38	36	30	24	22	16	15	15	16	14
24.0%	144	97	37	36	29	23	16	16	15	13	11	15

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.); SMEs (Sales < 5 Mio.); Mortgage; Revolving retail; Other retail

factor ρ. In buckets with small probabilities of default as well as low correlation factors, the idiosyncratic risk is relatively high, so that the portfolio must be substantially bigger to meet the target. This means that in the worst case, a portfolio must consist of at least 35,986 creditors to meet the assumptions of the ASRF framework at an accuracy of 5%. The same tendency can also be found for the target tolerance specification (4.50). We get critical numbers $I_{c,\text{abs}}^{(\text{ASRF})}$ ranging from 11 to 5,499 creditors (see Table 4.2), that are substantially lower compared to the critical numbers of the target tolerance. Thus, the critical number $I_{c,\text{abs}}^{(\text{fg})}$ is less conservative. This is caused by the effect that an increase of the confidence level for VaR calculations has a high impact, especially on risk buckets with low default rates. However, since for all those obligors the ASRF assumptions (see Sect. 2.6) still have to be valid, such big risk buckets may mainly be relevant for retail exposures in practice. Furthermore, it should be mentioned that these portfolio sizes are only valid for homogeneous portfolios. For heterogeneous portfolios, these numbers can be considerably higher, especially because the exposure weights differ between the obligors and thus concentration risk will occur.[196] In order to get an impression of real-world portfolio sizes, we refer to the data of the German credit register used in Düllmann and Erdelmeier (2009). The credit register contains all bank loans exceeding €1.5 million. In September 2006, out of 1,360 reporting financial enterprises,[197] there were in total 28 german banks which had at least 1,000 registered bank loans. Even if there are also smaller loans that are not included in the data, loans for corporate, sovereigns, and banks should mostly exceed the critical size. Hence, having a look at the required number of credits in Table 4.1, most bank portfolios cannot be treated as infinitely granular. Therefore, an improvement of measuring the portfolio-VaR is indeed advisable. However, it has to be mentioned that for portfolios with debtors incorporating low creditworthiness the ASRF solution is already sufficient for some hundred credits (or even less).

4.2.2.3 Probing First-Order Granularity Adjustment

After auditing the adequacy of the ASRF solution (b) compared to the discrete, "true" solution (a) in context of a homogeneous risk bucket, we now investigate the accuracy of the first order granularity adjustment (solution (c)). Similar to Sect. 4.2.2.2, we compare its accuracy with the discrete solution (a) but we additionally relate its result to the ASRF solution (b).

For the first (conservative) number $I_{c,\text{per}}^{(\text{1st Order Adj.})}$, we compare the analytically derived VaR including first order approximation (solution (c)) with the "true" VaR

[196] The case of heterogeneous portfolios will be analyzed in Sect. 4.2.2.5.

[197] Cf. Deutsche Bundesbank (2009).

of the discrete, binomial solution (a), both on a 0.999 confidence level. Again, we aim to meet a target tolerance of β and we get

$$I_{c,\text{per}}^{(\text{1st Order Adj.})} = \inf\left(n : \left|\frac{VaR_{0.999}^{(\text{1st Order Adj.})}\left(\tilde{L}\right)}{VaR_{0.999}^{(N)}\left(\tilde{L} = \frac{1}{N}\sum_{i=1}^{N} 1_{\{\tilde{D}_i\}}\right)} - 1\right| < \beta \ \forall N \in \mathbb{N}^{\geq n}\right), \text{ with } \beta = 0.05. \tag{4.51}$$

Thus, any analytically derived VaR of a risk bucket which includes more credits than $I_{c,\text{per}}^{(\text{1st Order Adj.})}$ does not differ from the "true" numerically derived VaR by more than 5%.

The results for $I_{c,\text{per}}^{(\text{1st Order Adj.})}$ for homogeneous risk buckets with a specific PD/ρ-combination are reported in Table 4.3. Obviously, the critical number varies from 7 to 6,100 credits. Compared to the ASRF solution (see Table 4.1 in Sect. 4.2.2.2), the critical values drop by 83.04% at a stretch. Precisely, we find that the number of credits that is necessary to ensure a good approximation of the "true" VaR is significantly lower with adjustment (c) than without adjustment (b). For example, a high quality retail portfolio (AAA) must consist of 5,027 compared to 26,051 credits if we neglect the first order adjustment. A medium quality corporate portfolio (BBB) must contain 106 compared to 442 credits. Thus, the minimum portfolio size should be small enough to hold for many real-world portfolios and we come to the conclusion that the first order adjustment works fine even with our conservative definition of a critical value.

Next, we relate the first order granularity adjustment (c) to the ASRF formula (b). We do that by defining a critical value $I_{c,\text{abs}}^{(\text{1st Order Adj.})}$ of credits similar to definition (4.50), but this time we proclaim that the VaR of the ASRF solution without first order granularity adjustment (b) at a confidence level of 0.999 should not exceed the VaR with first order granularity adjustment (c) at a confidence level of 0.995:

$$I_{c,\text{abs}}^{(\text{1st Order Adj.})} = \sup\left(n : VaR_{0.999}^{(\text{ASRF})}\left(\tilde{L}\right) < VaR_{0.995}^{(\text{1st Order Adj.})}\left(\tilde{L}\right)\right). \tag{4.52}$$

The confidence level of the ASRF solution is increased by a buffer of 4 basis points, which should incorporate the idiosyncratic risk of relatively fine-grained portfolios. If we use the first order granularity adjustment for approximating the true risk, the idiosyncratic risk of a portfolio with at $I_{c,\text{abs}}^{(\text{1st Order Adj.})}$ credits should already be included in the confidence level buffer.

The critical numbers of credits $I_{c,\text{abs}}^{(\text{1st Order Adj.})}$ are shown in Table 4.4. They contain a range from 14 to 5,170. It is interesting to note that these critical values do not differ widely from the numbers $I_{c,\text{abs}}^{(\text{fg})}$, where we compared the VaR of the ASRF solution (b) with the "true" VaR using the numerical, time-consuming discrete formula. Precisely, the average percentage difference between the critical

Table 4.3 Critical number of credits from that the first order adjustment can be stated to be sufficient for measuring the true VaR (see (4.51))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	6,100	4,227	879	833	693	519	337	228	152	89	63	42
3.5%	5,517	3,491	810	768	590	443	291	199	133	67	54	32
4.0%	5,027	3,192	688	653	503	413	251	174	127	60	49	28
4.5%	4,169	2,936	641	609	470	355	237	165	112	54	38	24
5.0%	3,846	2,456	546	519	401	334	205	132	107	45	37	22
5.5%	3,564	2,283	513	488	378	287	195	138	94	51	35	20
6.0%	3,317	2,129	484	460	358	272	169	121	83	46	33	20
6.5%	3,098	1,993	413	435	339	258	177	105	80	34	28	18
7.0%	2,902	1,872	392	373	322	246	154	111	77	40	29	18
7.5%	2,450	1,762	373	354	277	235	133	97	61	29	27	13
8.0%	2,309	1,494	355	338	264	203	128	84	59	35	25	16
8.5%	2,181	1,414	338	322	253	215	136	81	57	31	21	16
9.0%	2,065	1,341	323	308	242	186	118	79	55	23	23	16
9.5%	1,958	1,274	309	295	232	179	114	76	54	30	19	14
10.0%	1,861	1,212	266	253	199	172	110	74	58	22	20	14
10.5%	1,771	1,156	255	271	214	148	106	64	51	19	15	11
11.0%	1,689	1,103	245	234	206	143	92	62	44	23	15	11
11.5%	1,612	1,055	263	225	178	154	89	60	43	21	17	11
12.0%	1,541	1,010	227	217	171	133	86	52	51	18	19	11
12.5%	1,476	968	219	209	166	129	74	57	46	19	23	11
13.0%	1,414	928	211	202	160	125	81	49	40	15	12	12
13.5%	1,357	892	204	195	155	121	88	54	30	16	10	8
14.0%	1,303	858	197	188	167	117	68	41	34	17	8	8
14.5%	1,253	825	191	182	145	101	66	45	33	12	8	8
15.0%	1,206	795	185	176	141	110	64	56	28	14	15	8
15.5%	1,162	767	179	171	121	107	62	49	36	14	13	12
16.0%	1,120	740	154	166	118	104	69	37	31	16	13	9
16.5%	1,081	714	168	161	114	101	67	51	23	16	11	9
17.0%	1,044	690	145	156	125	87	58	35	30	9	11	9
17.5%	1,009	668	159	152	108	96	49	30	22	7	11	9
18.0%	976	646	154	131	105	83	55	39	18	7	9	9
18.5%	944	626	150	128	115	91	61	43	25	7	9	9
19.0%	914	606	146	124	112	79	53	28	21	13	9	9
19.5%	886	588	142	136	97	77	45	32	17	18	9	9
20.0%	859	570	123	118	95	75	44	36	20	14	9	9
20.5%	834	554	120	129	104	73	43	35	13	12	7	9
21.0%	809	538	117	112	90	63	42	30	16	10	7	9
21.5%	786	523	128	109	99	70	41	25	19	10	7	9
22.0%	764	508	111	106	86	77	51	29	22	8	7	9
22.5%	743	494	108	104	84	67	40	20	14	8	7	9
23.0%	722	481	119	114	92	57	39	36	11	8	7	9
23.5%	703	468	116	99	90	72	38	24	27	8	7	9
24.0%	684	456	101	97	88	55	32	16	18	8	7	9

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.); SMEs (Sales < 5 Mio.); Mortgage; Revolving retail; Other retail

Table 4.4 Critical number of credits from that the first order adjustment at confidence level 0.995 exceeds the infinite fine granularity at confidence level 0.999 (see (4.52))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	5,170	3,544	973	935	769	626	441	327	255	164	146	128
3.5%	4,029	2,773	774	744	615	501	356	265	209	136	122	109
4.0%	3,231	2,232	633	609	504	413	295	221	175	116	105	95
4.5%	2,650	1,836	528	508	422	347	249	188	150	101	91	85
5.0%	2,213	1,538	448	431	359	296	214	162	130	89	81	76
5.5%	1,875	1,307	385	371	310	256	186	142	114	79	72	69
6.0%	1,609	1,124	335	323	270	224	163	125	101	71	65	63
6.5%	1,395	977	295	284	238	198	145	112	91	64	60	59
7.0%	1,220	856	261	252	211	176	130	100	82	59	55	55
7.5%	1,075	757	233	225	189	158	117	91	74	54	50	51
8.0%	955	673	209	202	170	142	106	83	68	50	47	48
8.5%	853	602	189	182	154	129	96	75	62	46	44	45
9.0%	766	542	171	165	140	117	88	69	58	43	41	43
9.5%	691	490	156	151	128	108	81	64	53	40	38	41
10.0%	626	445	143	138	117	99	75	59	50	38	36	39
10.5%	570	405	131	127	108	91	69	55	46	36	34	37
11.0%	521	371	121	117	100	84	64	51	43	34	32	36
11.5%	477	340	112	108	92	78	60	48	40	32	31	34
12.0%	439	313	104	100	86	73	56	45	38	30	29	33
12.5%	404	289	96	93	80	68	52	42	36	29	28	32
13.0%	374	268	90	87	74	63	49	40	34	27	27	31
13.5%	346	248	84	81	70	59	46	37	32	26	26	30
14.0%	322	231	78	76	65	56	43	35	30	25	24	29
14.5%	299	215	74	71	61	52	41	33	29	24	24	28
15.0%	279	201	69	67	58	49	39	32	27	23	23	28
15.5%	261	188	65	63	54	47	36	30	26	22	22	27
16.0%	244	176	61	59	51	44	35	29	25	21	21	26
16.5%	229	165	58	56	48	42	33	27	24	20	20	26
17.0%	215	155	55	53	46	40	31	26	23	20	20	25
17.5%	202	146	52	50	43	38	30	25	22	19	19	25
18.0%	190	138	49	48	41	36	28	24	21	18	18	24
18.5%	180	130	46	45	39	34	27	23	20	18	18	24
19.0%	170	123	44	43	37	32	26	22	19	17	17	23
19.5%	160	116	42	41	36	31	25	21	19	17	17	23
20.0%	152	110	40	39	34	29	24	20	18	16	16	22
20.5%	144	105	38	37	32	28	23	19	17	16	16	22
21.0%	136	99	36	35	31	27	22	18	17	15	16	22
21.5%	129	94	35	34	29	26	21	18	16	15	15	22
22.0%	123	90	33	32	28	25	20	17	15	14	15	21
22.5%	117	85	32	31	27	24	19	17	15	14	15	21
23.0%	111	81	30	29	26	23	18	16	14	14	14	21
23.5%	106	78	29	28	25	22	18	15	14	13	14	21
24.0%	101	74	28	27	24	21	17	15	14	13	14	20

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.)
SMEs (Sales < 5 Mio.); Mortgage; Revolving retail; Other retail

numbers of Tables 4.2 and 4.4 is less than 10%. That means that the diversification behavior of the coarse grained solution and the first order approximation is very similar, i.e. the first order adjustment is a good approximation of the idiosyncratic risk of coarse grained portfolios.

4.2.2.4 Probing Second-Order Granularity Adjustment

Finally, we want to test the approximation if the (first- and) second-order adjustment is added to the ASRF formula, leading to solution (d). Similar to Sects. 4.2.2.2 and 4.2.2.3, we firstly examine the VaR according to this new formula (d) in comparison to the "exact" VaR from the coarse grained solution (a). Additionally, we analyze its performance with respect to the ASRF solution.

Again, we calculate a critical number $I_{c,\text{per}}^{(\text{1st + 2nd Order Adj.})}$ of credits to test the approximation accuracy with reference to the coarse grained formula (a) according to the "percentaged" accuracy with a target tolerance of 5% by

$$I_{c,\text{per}}^{(\text{1st + 2nd Order Adj.})} = \inf\left(n : \left| \frac{VaR_{0.999}^{(\text{1st + 2nd Order Adj.})}\left(\tilde{L}\right)}{VaR_{0.999}^{(N)}\left(\tilde{L} = \frac{1}{N}\sum_{i=1}^{N} 1_{\{\tilde{D}_i\}}\right)} - 1 \right| < \beta \ \forall N \in \mathbb{N}^{\geq n} \right),$$

$$\text{with} \quad \beta = 0.05, \tag{4.53}$$

using the (first- and) second-order adjustment as an approximation of the coarse-grained portfolio.

The results are presented in Table 4.5. Now, the critical number of credits ranges from 17 to 10,993. Compared to the ASRF solution (a), see Table 4.1 in Sect. 4.3.4.2, the necessary number of credits to meet the requirements can be reduced by 66.5% on average. Thus, the second-order adjustment is capable to detect idiosyncratic risk caused by a finite number of debtors to a certain extent. However, if we compare the results with the ones where only the first-order adjustment is used (see Table 4.3 in Sect. 4.3.4.3), the second-order adjustment performs worse.

We are able to verify this result by analyzing the second-order adjustment (d) in comparison to the exact ASRF solution (a). Therefore we introduce a critical number $I_{c,\text{abs}}^{(\text{1.+2. Order Adj.})}$ of credits, similar to the definition (4.52) in Sect. 4.3.4.3. We get

$$I_{c,\text{abs}}^{(\text{1st +2nd Order Adj.})} = \sup\left(n : VaR_{0.999}^{(\text{ASRF})}\left(\tilde{L}\right) < VaR_{0.995}^{(\text{1st +2nd Order Adj.})}\left(\tilde{L}\right)\right). \tag{4.54}$$

Thus, for each risk bucket with at least $I_{c,\text{abs}}^{(\text{1st +2nd Order Adj.})}$ credits the idiosyncratic risk, measured by the second-order adjustment on a confidence level 0.995, is included in the confidence level premium of 4 basis points of the ASRF solution (on a confidence level 0.999).

Table 4.5 Critical number of credits from that the first plus second order adjustment can be stated to be sufficient for measuring the true VaR (see (4.53))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	10,993	7,338	1,796	1,770	1,417	1,107	746	522	392	222	185	130
3.5%	9,309	6,251	1,503	1,427	1,150	941	620	440	327	193	163	115
4.0%	7,494	5,077	1,260	1,252	1,014	802	534	384	280	167	140	103
4.5%	6,405	4,367	1,109	1,054	858	683	460	323	255	148	120	90
5.0%	5,864	3,768	979	930	761	609	414	293	225	127	115	83
5.5%	5,056	3,256	866	824	677	544	373	266	199	118	103	78
6.0%	4,362	3,021	767	730	603	486	321	242	182	107	94	70
6.5%	4,055	2,622	680	647	537	435	304	210	167	100	86	64
7.0%	3,509	2,452	641	610	478	390	260	191	147	90	76	63
7.5%	3,286	2,132	570	542	453	349	248	183	141	84	74	60
8.0%	2,844	2,006	505	481	404	332	237	158	123	79	67	55
8.5%	2,679	1,892	480	457	385	297	214	160	119	71	63	51
9.0%	2,529	1,649	457	406	343	284	193	146	109	69	57	49
9.5%	2,394	1,563	406	387	328	254	174	133	105	67	58	51
10.0%	2,077	1,484	388	370	292	243	168	128	91	60	50	42
10.5%	1,974	1,412	344	354	280	234	161	116	88	56	49	43
11.0%	1,879	1,231	330	314	269	209	145	106	81	52	48	41
11.5%	1,791	1,175	316	302	239	201	140	109	88	51	45	38
12.0%	1,710	1,123	304	290	230	194	126	99	76	52	41	39
12.5%	1,484	1,075	269	257	222	173	131	96	74	51	42	37
13.0%	1,421	1,030	259	248	214	167	127	87	63	43	43	34
13.5%	1,362	897	250	239	190	149	106	79	70	42	37	34
14.0%	1,307	861	241	230	184	144	111	76	64	39	38	31
14.5%	1,256	828	233	203	177	139	92	80	54	38	34	32
15.0%	1,208	797	206	197	172	135	97	67	61	33	35	28
15.5%	1,163	768	199	190	152	131	94	65	52	39	31	29
16.0%	1,120	741	193	184	147	127	84	74	51	34	34	30
16.5%	1,081	715	187	178	143	113	89	67	46	38	30	26
17.0%	938	690	181	173	152	120	73	56	45	33	28	26
17.5%	906	600	176	168	135	106	71	64	51	31	26	27
18.0%	876	646	155	163	131	103	69	58	43	32	24	28
18.5%	847	562	150	144	115	101	74	52	42	30	27	23
19.0%	820	544	146	140	124	98	72	51	41	26	25	23
19.5%	795	527	142	150	109	86	64	45	37	29	23	24
20.0%	770	511	138	132	106	93	57	44	33	27	26	25
20.5%	747	496	134	115	93	91	67	43	42	23	21	26
21.0%	725	482	131	125	101	80	60	39	38	21	24	26
21.5%	704	468	114	122	88	78	53	42	31	24	22	20
22.0%	684	455	124	119	96	68	57	41	34	22	22	20
22.5%	665	442	121	116	94	67	56	44	39	22	20	21
23.0%	647	430	106	101	82	73	44	32	30	20	17	22
23.5%	629	419	103	99	80	64	43	35	24	18	21	22
24.0%	613	408	101	108	78	62	43	38	29	21	18	23

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.); SMEs (Sales < 5 Mio.); Mortgage; Revolving retail; Other retail

Table 4.6 Critical number of credits from that the first plus second order adjustment at confidence level 0.995 exceeds the infinite fine granularity at confidence level 0.999 (see (4.54))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	4,285	2,942	810	778	640	521	367	272	214	140	125	114
3.5%	3,266	2,254	633	609	503	411	292	218	173	115	104	97
4.0%	2,560	1,776	508	489	406	333	238	180	143	97	89	84
4.5%	2,050	1,429	417	401	334	275	198	151	121	83	77	75
5.0%	1,671	1,170	347	335	279	231	168	128	103	73	68	67
5.5%	1,380	971	294	283	237	196	144	111	90	64	60	61
6.0%	1,153	815	251	242	203	169	124	96	79	57	54	56
6.5%	973	691	216	209	176	147	109	85	70	52	49	51
7.0%	827	590	188	182	153	128	96	75	62	47	44	48
7.5%	708	507	164	159	135	113	85	67	56	43	41	44
8.0%	610	439	145	140	119	100	76	60	50	39	38	42
8.5%	527	382	128	124	106	89	68	54	46	36	35	39
9.0%	458	333	114	110	94	80	61	49	42	33	32	37
9.5%	399	292	102	98	84	72	55	45	38	31	30	35
10.0%	349	257	91	88	76	65	50	41	35	29	28	33
10.5%	306	226	82	79	68	59	46	37	32	27	27	32
11.0%	268	200	74	72	62	53	42	34	30	25	25	31
11.5%	264	177	67	65	56	48	38	32	28	24	24	29
12.0%	271	156	60	59	51	44	35	29	26	22	22	28
12.5%	266	173	55	53	46	40	32	27	24	21	21	27
13.0%	257	172	50	48	42	37	30	25	22	20	20	26
13.5%	248	167	45	44	39	34	27	23	21	19	19	25
14.0%	238	162	41	40	36	31	25	22	20	18	18	24
14.5%	229	156	38	37	33	29	24	20	18	17	18	24
15.0%	219	150	34	34	30	26	22	19	17	16	17	23
15.5%	210	144	38	36	27	24	20	18	16	15	16	22
16.0%	201	139	38	36	28	23	19	17	15	15	15	22
16.5%	193	133	37	36	29	21	18	16	14	14	15	21
17.0%	185	128	37	35	29	22	16	15	14	13	14	21
17.5%	177	123	36	34	28	23	15	14	13	13	14	20
18.0%	170	118	35	33	28	23	14	13	12	12	13	20
18.5%	163	113	34	33	27	22	13	12	12	12	13	19
19.0%	156	109	33	32	26	22	15	11	11	11	12	19
19.5%	150	105	32	31	26	21	15	11	10	11	12	19
20.0%	145	101	31	30	25	21	15	10	10	11	12	18
20.5%	139	97	30	29	24	20	15	10	9	10	11	18
21.0%	134	94	29	28	24	20	14	9	9	10	11	18
21.5%	129	90	28	27	23	19	14	10	8	10	11	17
22.0%	124	87	27	26	22	19	14	10	8	9	10	17
22.5%	120	84	26	26	22	18	14	10	8	9	10	17
23.0%	115	81	26	25	21	18	13	10	7	9	10	16
23.5%	111	78	25	24	20	17	13	10	7	8	9	16
24.0%	108	75	24	23	20	17	13	10	7	8	9	16

Corporates, sovereigns, and banks   SMEs (5Mio. < Sales < 50 Mio.)
SMEs (Sales < 5 Mio.)   Mortgage   Revolving retail   Other retail

The critical numbers presented in Table 4.6 range from 7 to 4,285. Obviously, these results are considerably higher than those of Table 4.4 and therefore the predefined target value of accuracy is reached with lower numbers of credits. Thus, the idiosyncratic risk is underestimated with the second order adjustment compared to the numerical "true" solution (a) (see the results in Sect. 4.2.2.2) and is not measured with such a high accuracy as the first order adjustment does (see Sect. 4.2.2.3). Concretely, this value is reduced by averaged 32.7% credits.

To conclude, the second-order adjustment (d) converges faster to the asymptotic value of the ASRF solution (b), which confirms the findings of Sect. 4.2.2.1. A possible reason is that the VaR measure using the first order approximation may be "corrected" into the direction of the ASRF solution by incorporating the second order adjustment. The possibility of this behavior is given due to the alternating sign in the derivatives of VaR; see (4.31).[198] Thus, taking more derivatives into account could solve the problem but would lead to even more uncomfortable equations.[199] Despite these theoretical questions, it can be stated that in homogeneous portfolios, an excellent approximation of the true VaR can be achieved with the granularity adjustment.

4.2.2.5 Probing Granularity for Inhomogeneous Portfolios

The previous analyses showed that the granularity adjustment works fine for homogeneous portfolios. In this section, we test if the approximation accuracy of the presented general formulas will hold for portfolios consisting of loans with different exposures and credit qualities. This means that the credits in the portfolio vary in exposure weight and in probability of default, and we analyze if the portfolio loss for coarse grained portfolios could still be quantified satisfactorily by the granularity adjustment.

Concretely, we examine high quality portfolios with probabilities of default ranging from 0.02 to 0.79% and lower quality portfolios with probabilities of default ranging from 0.2 to 7.9%. Additionally, we define a basic risk bucket consisting of 20 loans with exposures between €35 and 200 million.[200] In order to measure the portfolio size with respect to concentration risk, we use the effective number of loans n^* (see (2.87)), rather than the number of loans n. Consequently, this effective number is more than 25% below the true number of credits.

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[198] This is true not only for the first five derivatives but also for all following derivatives; see the general formula for all derivatives of VaR in (4.213).

[199] However, we also have to take into consideration that the Taylor series is potentially not convergent at all or does not converge to the correct value. For a further discussion see Martin and Wilde (2002) and Wilde (2003).

[200] The used portfolio is based on Overbeck (2000), see also Overbeck and Stahl (2003), but reduced to 20 loans to achieve more test portfolios with a small number of credits.

A variation of portfolio size is reached by reproducing the loans of the basic risk bucket so that portfolios with 40, 60, ..., 400, 800, 1,600 and 4,000 loans result. Using an asset correlation $\rho = 20\%$ and a confidence level of 0.999, we compute the granularity add-on with the presented first-order and second-order adjustment. Because the exact value cannot be determined analytically for heterogeneous portfolios, we compute the "true" VaR with Monte Carlo simulations using three million trials.[201] Finally, we compare this "true" VaR with the ASRF solution, so that we receive the granularity add-on.

The simulated results for the granularity add-on for high and low quality portfolios are presented in Fig. 4.3 (see the circles and dots). Therefore, the add-on for the minimum size of 40 loans with $1/n^* \approx 0.035$ is 5.0% (6.2%) for the high (low) quality portfolio. This is equal to a relative correction of +112.5% (+30.5%) compared to a hypothetical infinitely fine grained portfolio. This shows again the relatively high impact of idiosyncratic risk in small high quality portfolios. With shifting to bigger sized portfolios, the effective number of credits shifts to zero and

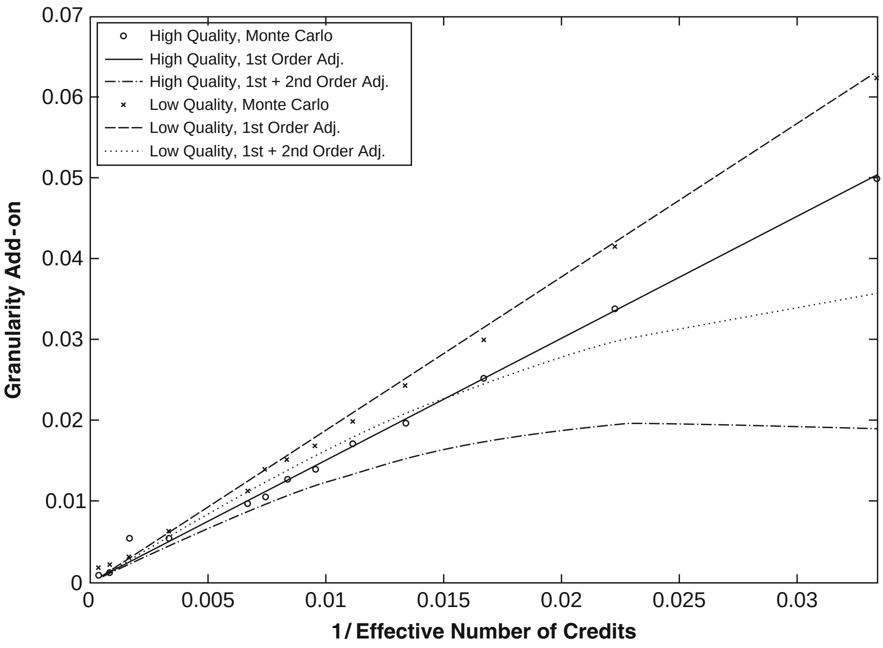

Fig. 4.3 Granularity add-on for heterogeneous portfolios calculated analytically with first-order (*solid lines*) and second-order (*dotted lines*) adjustments as well as with Monte Carlo simulations (+ and o) using three million trials

[201]Due to the high number of trials, which corresponds to 3,000 hits in the tail for a confidence level of 0.999, the simulation noise should be negligible.

the granularity add-on decreases almost exactly linear in terms of $1/n^*$ – even for high quality portfolios. This result is contrary to Gordy (2003), who exhibits a concave characteristic of the granularity add-on. This might be due to the fact that Gordy (2003) uses a CreditRisk$^+$ framework, whereas we analyze the effect of the granularity with the CreditMetrics one-factor model that is consistent with the Basel II assumptions. Summing up, the granularity add-on in Fig. 4.3 can be approximated with a linear function. Indeed, the (linear) first order adjustment is a very good approximation for heterogeneous portfolios of high as well as low quality. Just like in the previous sections, the second-order adjustment leads to a reduction of the granularity add-on. Thus, it can be characterized as less conservative, but comparing the results we strongly recommend the first-order adjustment.

4.3 Measurement of Name Concentration Using the Risk Measure Expected Shortfall

4.3.1 Adjusting for Coherency by Parameterization of the Confidence Level

As shown in Sect. 2.2.3, the commonly used VaR is not coherent because it is not necessarily subadditive. As long as we stay in the ASRF framework, this characteristic is not problematic because in this context, the VaR is exactly additive.[202] However, if we leave the ASRF framework, this behavior is not guaranteed anymore.[203] Nevertheless, many contributions that deal with concentration risk in the context of Basel II use the VaR to quantify credit risk without questioning the risk measure (possibly to be consistent with the ASRF framework), even if the subadditivity could get problematic if concentration risk is considered.[204] Thus, it could be beneficial to change the measure of risk, e.g. to use the coherent Expected Shortfall (ES). However, we cannot simply replace the VaR with the ES since the resulting difference in the capital requirements would not only stem from a more convenient measurement of concentration risk but also from the fact that the ES exceeds the VaR by definition. Against this background, we propose a procedure how the ES can be used instead of the VaR for the measurement of credit risk by accurately choosing a different confidence level. Based on this result, we analyze the performance of the ASRF formula, the first-order, and the second-order granularity adjustment when the ES is used instead of the VaR in Sect. 4.3.4 after deriving both adjustment formulas in Sect. 4.3.2.

[202] Cf. Sect. 2.6.

[203] This is true for a violation of both the granularity and the single risk factor assumption.

[204] See e.g. Heitfield et al. (2006), Cespedes et al. (2006), Düllmann (2006), as well as Düllmann and Masschelein (2007).

Before we change the risk measure, we will study the characteristics of the VaR for credit portfolios and analyze the need for using the ES. For our analyses, we continue to omit the first assumption of the ASRF framework leading to a finite granularity and calculate the VaR as well as the ES within the binomial model of Vasicek and the ASRF framework.

We start with computing the VaR at a confidence level $\alpha = 0.999$ for non-asymptotic portfolios with $PD = 0.5\%$ and $\rho = 20\%$. In Fig. 4.4, the VaR for the ASRF framework and for the Vasicek binomial model is plotted in the cases of $n = 1$ to $n = 300$ homogeneous credits. The VaR for an infinite number of credits is 9.1%. For a finite number of credits, the risk is higher because the unsystematic risk cannot be diversified. The problem is that the risk should be monotonously decreasing with a higher number of credits ("monotonicity of specific risk-property"[205]) but this behavior is not reflected by the VaR as a risk measure. Instead, we find that the VaR follows a downward sloping "saw-toothed" pattern. Although the sub-additivity axiom is not violated in the example, it is obvious that the measured risk should not increase with a higher number of credits and thus a better diversification. It is also possible to construct superadditive examples with a different parameter setting but this example gives a clear demonstration that it is problematic to use the VaR if there is concentration risk such as name concentration.

The saw-toothed pattern can also be explained intuitively: In the 99.9% worst-case scenario one credit out of 1, 2, 3, 4, or 5 credits defaults, which leads to a VaR of 1, 1/2, 1/3, 1/4, or 1/5. If the size of the portfolio is increased further, one additional credit defaults in the 99.9% scenario. Thus, the VaR increases from $1/5 = 20\%$ to $2/6 = 33.\bar{3}\%$. If additional credits are added to the portfolio, the

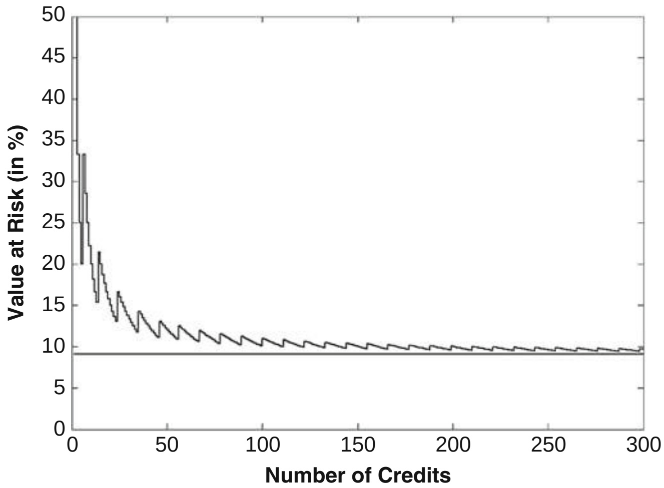

Fig. 4.4 Value at Risk in the ASRF and the Vasicek model

[205] See Albanese and Lawi (2004), p. 215, for this property of a reasonable risk measure.

VaR will increase until a third credit defaults in the considered 99.9% scenario, and so on. From a probabilistic perspective, the demonstrated problems are mainly a result of the deviation for discrete distributions $\mathbb{P}[\tilde{L} \leq VaR_\alpha(\tilde{L})] - \alpha > 0$, which is mostly decreasing with additional credits but jumps to a higher value when the difference would (theoretically) go below zero.[206] Against this background, it could be tried to define the VaR differently from the common definition of the (lower) VaR (2.12). Also the upper VaR definition (2.13) does not solve the problem. However, if the VaR was defined as the maximal loss in the best $100 \cdot \alpha\%$ scenarios

$$VaR_\alpha^{(-)}\left(\tilde{L}\right) = \sup\left\{l \in \mathbb{R} \,|\, \mathbb{P}\left[\tilde{L} \leq l\right] < \alpha\right\} \tag{4.55}$$

instead of the minimal loss in the worst $100 \cdot (1 - \alpha)\%$, we have the contrary case of a negative deviation $\mathbb{P}\left[\tilde{L} \leq VaR_\alpha^{(-)}\right] - \alpha < 0$. If we rewrite the common VaR definition as

$$VaR_\alpha^{(+)}\left(\tilde{L}\right) = \inf\left\{l \in \mathbb{R} \,|\, \mathbb{P}\left[\tilde{L} \leq l\right] \geq \alpha\right\} = \sup\left\{l \in \mathbb{R} \,|\, \mathbb{P}\left[\tilde{L} < l\right] < \alpha\right\}, \tag{4.56}$$

it is obvious to see that the VaR from definition (4.55) is always below the VaR from definition (4.56). In the considered case of n homogeneous credits the difference between both definitions always equals[207]

$$VaR_\alpha^{(+)} - VaR_\alpha^{(-)} = \frac{1}{n}. \tag{4.57}$$

As the positive deviation $p^{(+)} := \mathbb{P}\left[\tilde{L} \leq VaR_\alpha^{(+)}\right] - \alpha > 0$ is high when the negative deviation $p^{(-)} := \mathbb{P}\left[\tilde{L} \leq VaR_\alpha^{(-)}\right] - \alpha < 0$ is small, we could define an *interpolated Value at Risk* $VaR^{(\mathrm{int})}$ as follows:

$$\begin{aligned} VaR_\alpha^{(\mathrm{int})} &= \frac{\mathbb{P}\left[\tilde{L} \leq VaR_\alpha^{(+)}\right] - \alpha}{\mathbb{P}\left[\tilde{L} \leq VaR_\alpha^{(+)}\right] - \mathbb{P}\left[\tilde{L} \leq VaR_\alpha^{(-)}\right]} VaR_\alpha^{(-)} \\ &\quad + \frac{\alpha - \mathbb{P}\left[\tilde{L} \leq VaR_\alpha^{(+)}\right]}{\mathbb{P}\left[\tilde{L} \leq VaR_\alpha^{(+)}\right] - \mathbb{P}\left[\tilde{L} \leq VaR_\alpha^{(-)}\right]} VaR_\alpha^{(+)} \\ &= \frac{p^{(+)}}{p^{(+)} - p^{(-)}} VaR_\alpha^{(-)} - \frac{p^{(-)}}{p^{(+)} - p^{(-)}} VaR_\alpha^{(+)}. \end{aligned} \tag{4.58}$$

[206] Of course the definition of the VaR does not allow a negative deviation and the VaR jumps to a higher value instead.

[207] See Appendix 4.5.11.

In Fig. 4.5, this interpolated VaR as well as $VaR_\alpha^{(+)}$, $VaR_\alpha^{(-)}$ and the ASRF solution are plotted. We find that the saw-toothed pattern, which is contradictory to the "monotonicity of specific risk-property", almost vanishes for the interpolated VaR, especially if we do not consider a very small number of credits. Thus, against the background of name concentration risk, definition (4.58) seems to be much less problematic than the common VaR definition (4.56).

For comparison, we also compute the ES for the identical portfolio setting. For calculation of the ES within the Vasicek model, we have to apply (2.76). The ES in the Basel II framework can be calculated with[208]

$$ES_\alpha^{(\text{Basel})}\left(\tilde{L}\right) = \frac{1}{1-\alpha}\sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \Phi_2\left(-\Phi^{-1}(\alpha), \Phi^{-1}(PD_i), \sqrt{\rho_i}\right), \quad (4.59)$$

which is based on the identity (2.93) of the ES within the ASRF framework and the conditional PD of the Vasicek model. Thus, (4.59) relies on the same assumptions as the Basel II formula (2.97) but uses the ES instead of the VaR for measuring the risk. As illustrated in Fig. 4.6, the ES satisfies the "monotonicity of specific

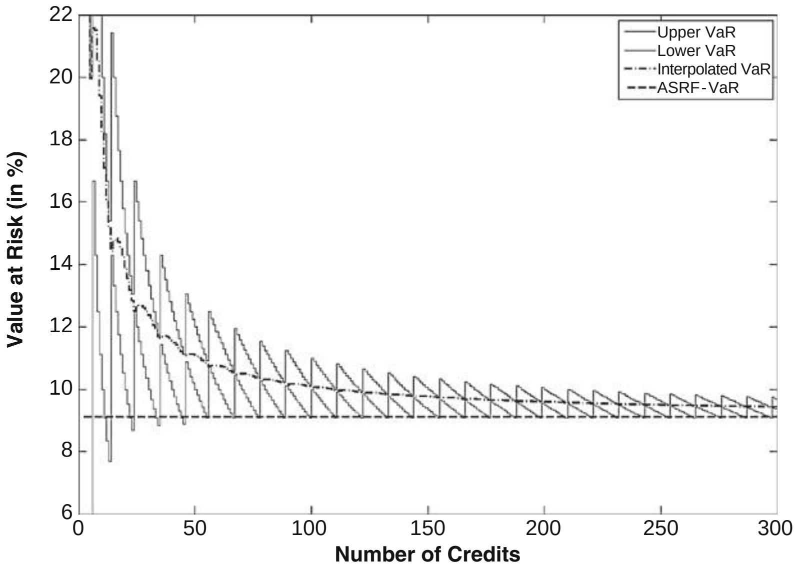

Fig. 4.5 Different Value at Risk measures in the Vasicek model

[208]See Appendix 4.5.12.

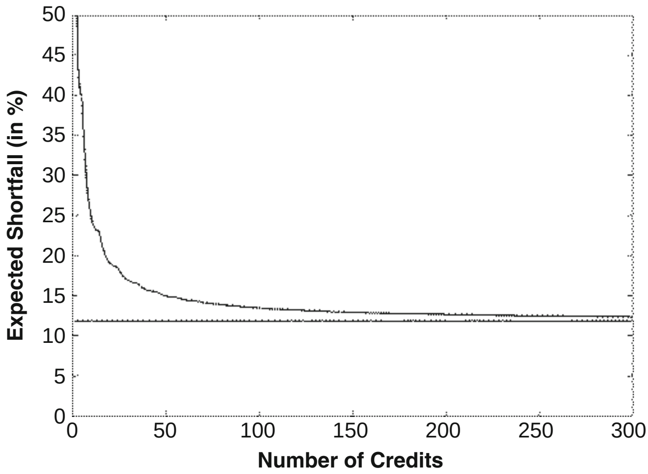

Fig. 4.6 Expected Shortfall in the ASRF and the Vasicek model

risk-property". This is one relevant advantage compared to the VaR, even if the VaR definition (4.58) is applied. Although this new VaR definition is already an improvement compared to the common definition, there are still some (minor) violations of the "monotonicity of specific risk-property", and the lack of subadditivity is still existent. Against this background, it could be beneficial to change the risk measure from VaR to ES if the portfolio contains concentration risk.[209] However, the measured economic capital would be significantly higher if it is determined on the basis of the ES instead of the VaR (by the use of the same confidence level), what is not the intended consequence of the change of the risk measure. In our example even the ASRF solution rises from 9.1% to 11.81%. Instead, we would only like to use the appreciated properties for concentration risk without being bound to increase the amount of economic capital. Therefore, the confidence level will be adjusted as described subsequently.

If we change the risk measure, we have to ensure that the new risk measure (the ES), on the one hand, is consistent with the framework presented in Pillar 2 of Basel II to get meaningful results for additional capital requirements stemming from concentration risk. On the other hand, the new risk measure should still match the capital requirements of Pillar 1 if the portfolio under consideration fulfills the assumptions of the ASRF framework; i.e. in the context of the ASRF framework, the capital requirements should not differ, regardless of whether the risk is measured by the VaR or by the ES. Therefore, we examine the VaR at the given

[209] As mentioned in Sect. 2.6, the VaR is exactly additive and therefore unproblematic in the context of the ASRF framework.

Table 4.7 Confidence level for the ES so that the ES is matched with the VaR with confidence level 0.999 for portfolios of different quality

Portfolio type/quality	$VaR_{0.999}$ and ES_{α} (%)	Confidence level α (ES) (%)
(I) AAA only	0.57	99.672
(II) Very high	6.12	99.709
(III) High	7.59	99.711
(IV) Average	12.94	99.719
(V) Low	20.89	99.726
(VI) Very low	23.30	99.727
(VII) CCC only	57.00	99.741

confidence level 0.999 for several (infinitely granular) bank portfolios of different quality. As a next step, we determine the confidence level of the ES that is necessary to match the results for both risk measures. We define this ES-confidence level α $(= \alpha(ES))$ implicitly as

$$ES_{\alpha}^{(\text{Basel})}\left(\tilde{L}\right) = VaR_{0.999}^{(\text{Basel})}\left(\tilde{L}\right), \tag{4.60}$$

with $VaR_{0.999}^{(\text{Basel})}$ given by (2.97) and $ES_{\alpha}^{(\text{Basel})}$ presented in (4.59).

Firstly, we investigate the extreme cases that all creditors of a bank have a rating of (I) AAA or (VII) CCC.[210] As can be seen in Table 4.7, the ES-confidence level must be in a range between 99.67% and 99.74%. Using these confidence levels, the economic capital is almost identical, regardless of whether the VaR or the ES is used.

Additionally, we use five portfolios with different credit quality distributions (very high, high, average, low, and very low) that are visualized in Fig. 4.7.[211] All resulting confidence levels are between 99.71% and 99.73% with mean 99.72%. Even if there is some interconnection between the confidence level and the portfolio quality, an ES-confidence level of $\alpha = 99.72\%$ seems to be accurate for most real-world portfolios.

4.3.2 *Considering Name Concentration with the Granularity Adjustment*

4.3.2.1 First-Order Granularity Adjustment for One-Factor Models

As argued in Sect. 4.3.1, the VaR can be a problematic risk measure if the assumptions of the ASRF framework, which includes the infinite granularity assumption (A)

[210] We use the idealized default rates from Standard and Poors, see Brand and Bahar (2001), ranging from 0.01% to 18.27%, but the results do not differ widely for different values.

[211] The portfolios with high, average, low, and very low quality are taken from Gordy (2000). We added a portfolio with very high quality.

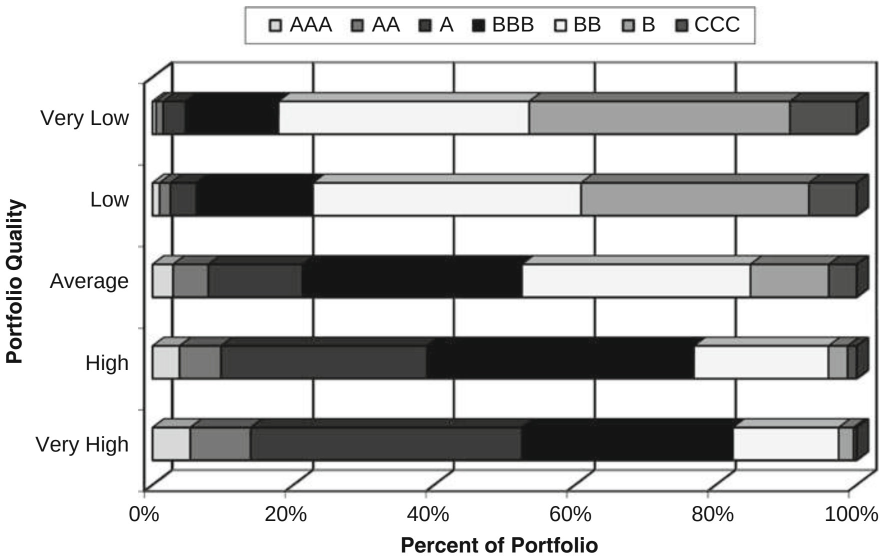

Fig. 4.7 Portfolio quality distributions

of Sect. 2.6, are not fulfilled anymore. Based on the methodology of Sect. 4.3.1, we know which confidence level is adequate if credit risk and especially concentration risk is measured on the basis of the more convenient ES instead of the VaR. However, the approximation formulas of Sect. 4.2.1 are only valid for the VaR. Thus, the ES-based granularity adjustment formulas will be derived subsequently. While the first-order granularity adjustment is already known in the literature, the second-order adjustment is a new result. The principle behind the granularity adjustment remains unchanged, regardless of whether the VaR or the ES is used as the risk measure. Thus, using the abbreviation

$$\tilde{L} = \mathbb{E}\big(\tilde{L}\,|\,\tilde{x}\big) + \big[\tilde{L} - \mathbb{E}\big(\tilde{L}\,|\,\tilde{x}\big)\big] =: \tilde{Y} + \lambda\tilde{Z}, \tag{4.61}$$

we perform a Taylor-series expansion around the systematic loss at $\lambda = 0$, leading to

$$\begin{aligned} ES_\alpha\big(\tilde{L}\big) &= ES_\alpha\big(\tilde{Y} + \lambda\tilde{Z}\big) \\ &= ES_\alpha\big(\tilde{Y}\big) + \lambda\left[\frac{dES_\alpha\big(\tilde{Y} + \lambda\tilde{Z}\big)}{d\lambda}\right]_{\lambda=0} + \frac{\lambda^2}{2!}\left[\frac{d^2ES_\alpha\big(\tilde{Y} + \lambda\tilde{Z}\big)}{d\lambda^2}\right]_{\lambda=0} \\ &\quad + \cdots + \frac{\lambda^m}{m!}\left[\frac{d^mES_\alpha\big(\tilde{Y} + \lambda\tilde{Z}\big)}{d\lambda^m}\right]_{\lambda=0} + \cdots. \end{aligned} \tag{4.62}$$

According to Sect. 4.2.1.1, the first-order adjustment can be calculated as the Taylor series expansion up to the quadratic term. With respect to Wilde (2003) and Rau-Bredow (2004), the needed first and second derivative of ES are given as[212]

$$\left.\frac{dES_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}{d\lambda}\right|_{\lambda=0}=\mathbb{E}\left[\tilde{Z}\,|\,\tilde{Y}>q_\alpha\left(\tilde{Y}\right)\right], \tag{4.63}$$

$$\left.\frac{d^2ES_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}{d^2\lambda}\right|_{\lambda=0}=\frac{f_Y\left(q_\alpha\left(\tilde{Y}\right)\right)\mathbb{V}\left[\tilde{Z}\,|\,\tilde{Y}=q_\alpha\left(\tilde{Y}\right)\right]}{1-\alpha}. \tag{4.64}$$

Similar to the VaR, the first derivative is zero:

$$\begin{aligned}\mathbb{E}\left[\tilde{Z}\,|\,\tilde{Y}>q_\alpha\left(\tilde{Y}\right)\right]&=\frac{1}{\lambda}\cdot\mathbb{E}\left[\tilde{L}-\mathbb{E}\left(\tilde{L}\,|\,\tilde{x}\right)|\tilde{Y}>q_\alpha\left(\tilde{Y}\right)\right]\\&=\frac{1}{\lambda}\cdot\mathbb{E}\left[\tilde{L}\,|\,\tilde{Y}>q_\alpha\left(\tilde{Y}\right)\right]-\frac{1}{\lambda}\cdot\mathbb{E}\left[\tilde{L}\,|\,\tilde{Y}>q_\alpha\left(\tilde{Y}\right)\right]=0.\end{aligned} \tag{4.65}$$

With

$$\begin{aligned}&\tilde{Y}=q_\alpha\left(\tilde{Y}\right)\\&\Leftrightarrow\tilde{x}=q_{1-\alpha}(\tilde{x})\end{aligned} \tag{4.66}$$

and

$$\lambda^2\cdot\mathbb{V}\left[\tilde{Z}\,|\,\tilde{Y}\right]=\mathbb{V}\left[\lambda\tilde{Z}\,|\,\tilde{Y}\right]=\mathbb{V}\left[\tilde{L}-\tilde{Y}\,|\,\tilde{Y}\right]=\mathbb{V}\left[\tilde{L}\,|\,\tilde{Y}\right], \tag{4.67}$$

the quadratic term of the Taylor series expansion (4.62) is equivalent to

$$\begin{aligned}\Delta l_1&=\frac{\lambda^2}{2}\left(\frac{f_Y\left(q_\alpha\left(\tilde{Y}\right)\right)\mathbb{V}\left[\tilde{Z}\,|\,\tilde{Y}=q_\alpha\left(\tilde{Y}\right)\right]}{1-\alpha}\right)\\&=-\frac{1}{2}\frac{f_Y\left(q_\alpha\left(\tilde{Y}\right)\right)\mathbb{V}\left[\tilde{L}\,|\,\tilde{x}=q_{1-\alpha}(\tilde{x})\right]}{1-\alpha}.\end{aligned} \tag{4.68}$$

Using[213]

$$f_Y(y)=-f_x(x)\frac{1}{dy/dx}, \tag{4.69}$$

[212] The derivatives of ES are derived in Appendix 4.5.13 and 4.5.14.

[213] Cf. (4.8).

the first-order granularity adjustment results in

$$ES_\alpha^{(n)} \approx ES_\alpha^{(\text{ASRF})} + \Delta l_1 =: ES_\alpha^{(\text{1st Order Adj.})}$$
$$\text{with } \Delta l_1 = -\frac{1}{2(1-\alpha)} \frac{f_x(x)\,\mathbb{V}\left[\tilde{L}\,|\,\tilde{x} = q_{1-\alpha}(\tilde{x})\right]}{\frac{d}{dx}\mathbb{E}\left[\tilde{L}\,|\,\tilde{x} = x\right]\big|_{x=q_{1-\alpha}(\tilde{x})}}. \tag{4.70}$$

Analogous to the VaR-based first-order adjustment, the ES-based term Δl_1 is linear in terms of $1/n$, which means that the measured idiosyncratic risk component is halved if the number of credits is doubled. Furthermore, the adjustment formula takes the conditional variance into consideration but neglects all higher conditional moments. Thus, incorporating the add-on formula (4.70) leads to a reduction of the error from $O(1/n)$ to $O(1/n^2)$.

4.3.2.2 First-Order Granularity Adjustment for the Vasicek Model

It is straightforward to calculate the ES-based granularity adjustment for the Vasicek model. This means that the conditional PD is assumed to be given by

$$p_i(x) = \Phi\left(\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i}\cdot x}{\sqrt{1-\rho_i}}\right) \tag{4.71}$$

and the systematic factor is standard normally distributed, which is analogous to Sect. 4.2.1.2. If we want to calculate the granularity adjustment (4.70), we can use the expression for the conditional variance and the derivative of the conditional expectation $d\mu_{1,c}/dx$ from Sect. 4.2.1.2. This directly leads to the formula for the ES-based granularity adjustment within the Vasicek model:

$$\begin{aligned}\Delta l_1 &= -\frac{1}{2(1-\alpha)} \frac{\varphi\eta_{2,c}}{d\mu_{1,c}/dx}\bigg|_{x=\Phi^{-1}(1-\alpha)} \\ &= \frac{\varphi\left(\Phi^{-1}(1-\alpha)\right)}{2(1-\alpha)} \frac{\sum_{i=1}^{n} w_i^2 \cdot \left[\left(ELGD_i^2 + VLGD_i\right)\cdot\Phi(z_i) - ELGD_i^2\cdot\Phi^2(z_i)\right]}{\sum_{i=1}^{n} w_i\cdot ELGD_i\cdot\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\cdot\varphi(z_i)},\end{aligned} \tag{4.72}$$

with $z_i = \frac{\Phi^{-1}(PD_i)+\sqrt{\rho_i}\Phi^{-1}(\alpha)}{\sqrt{1-\rho_i}}$, which can be simplified for homogeneous portfolios to

$$\Delta l_1 = \frac{1}{2n}\frac{\varphi\left(\Phi^{-1}(1-\alpha)\right)}{(1-\alpha)}\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\frac{\Phi(z)}{\varphi(z)}\left(\frac{ELGD^2+VLGD}{ELGD} - ELGD\cdot\Phi(z)\right), \tag{4.73}$$

with $z = \frac{\Phi^{-1}(PD)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}$.

4.3.2.3 Second-Order Granularity Adjustment for One-Factor Models

In order to reduce the approximation error for portfolios consisting of a small number of credits, additional elements of the Taylor-series expansion (4.62) will be calculated and analyzed subsequently. Thus, we derive all terms of order $O(1/n^2)$, which is analogous to Sect. 4.3.2.3 for the VaR-based granularity adjustment. As a consequence, not only the conditional variance but also the conditional skewness is taken into account. The resulting expression for the ASRF solution including the second-order granularity adjustment Δl_2 is

$$VaR_\alpha^{(\text{1st + 2nd Order Adj.})} = VaR_\alpha^{(\text{ASRF})} + \Delta l_1 + \Delta l_2, \tag{4.74}$$

where Δl_2 represents the $O(1/n^2)$ elements of (4.62). We already know from Appendix 4.5.8 that the third and a part of the fourth element of the Taylor series are the relevant terms for the second-order adjustment.[214] As can immediately be seen from the Taylor series expansion (4.62), the third and the fourth derivatives of ES are required for the calculation of the additional terms. Based on the formula for all derivatives of VaR, it is possible to determine a formula for arbitrary derivatives of ES. This general formula is derived in Appendix 4.5.13,[215] but for our purposes it is sufficient to use a formula for the first five derivatives of ES:[216]

$$\begin{aligned} \left.\frac{d^m ES_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}{d\lambda^m}\right|_{\lambda=0} &= (-1)^m \cdot \frac{1}{1-\alpha} \cdot \Bigg(\frac{d^{m-2}\left(\mu_m\left(\tilde{Z}\,|\,\tilde{Y}=y\right)f_Y(y)\right)}{dy^{m-2}} \\ &\quad \left. - \kappa(m)\cdot\left[\frac{1}{f_Y(y)}\cdot\frac{d\left(\mu_2\left(\tilde{Z}\,|\,\tilde{Y}=y\right)f_Y(y)\right)}{dy}\cdot\frac{d^{m-3}\left(\mu_{m-2}\left(\tilde{Z}\,|\,\tilde{Y}=y\right)f_Y(y)\right)}{dy^{m-3}}\right]\Bigg)\right|_{y=q_\alpha(\tilde{Y})}, \end{aligned} \tag{4.75}$$

with $\kappa(1) = \kappa(2) = 0,\ \kappa(3) = 1,\ \kappa(4) = 3$, and $\kappa(5) = 10$.

With these derivatives and due to

$$\lambda^m \cdot \mu_m\left(\tilde{Z}\,|\,\tilde{Y}=y\right)\big|_{y=q_\alpha(\tilde{Y})} = \eta_m\left[\tilde{L}\,|\,\tilde{Y}=y\right]\big|_{y=q_\alpha(\tilde{Y})} =: \eta_m(y)\big|_{y=q_\alpha(\tilde{Y})}, \tag{4.76}$$

[214] The explanations regarding the order of the derivatives of VaR in Appendix 4.5.8 are valid for the derivatives of ES, too.

[215] See also Wilde (2003).

[216] See Appendix 4.5.14.

the second-order adjustment for one-factor models is given as

$$
\begin{aligned}
\Delta l_2 &= \frac{(-1)^3}{3!}\,\frac{1}{1-\alpha}\left[\frac{d(\eta_3(y)f_Y(y))}{dy}\right] \\
&\quad + \frac{(-1)^4}{4!}\,\frac{1}{1-\alpha}\left[-3\left(\frac{1}{f_Y(y)}\cdot\frac{d(\eta_2(y)f_Y(y))}{dy}\cdot\frac{d(\eta_2(y)f_Y(y))}{dy}\right)\right]\Bigg|_{y=q_\alpha(\tilde{Y})} \\
&= -\frac{1}{6(1-\alpha)}\left[\frac{d}{dy}(\eta_3(y)f_Y(y))\right] - \frac{1}{8(1-\alpha)}\,\frac{1}{f_Y(y)}\left[\frac{d}{dy}(\eta_2(y)f_Y(y))\right]^2\Bigg|_{y=q_\alpha(\tilde{Y})}.
\end{aligned}
\tag{4.77}
$$

Using $f_Y = -\frac{f_x}{dy/dx}$ and recalling that $\eta_m(y)\big|_{y=q_\alpha(\tilde{Y})} = \eta_m(\tilde{L}\,|\,\tilde{x}=x)\big|_{x=q_{1-\alpha}(\tilde{x})} =: \eta_{m,c}\big|_{x=q_{1-\alpha}(\tilde{x})}$ (cf. (4.9)), this leads to

$$
\begin{aligned}
\Delta l_2 &= \frac{1}{6(1-\alpha)}\,\frac{1}{dy/dx}\,\frac{d}{dx}\left(\frac{\eta_{3,c}f_x}{dy/dx}\right) \\
&\quad + \frac{1}{8(1-\alpha)}\,\frac{dy/dx}{f_x}\left[\frac{1}{dy/dx}\,\frac{d}{dx}\left(\frac{\eta_{2,c}f_x}{dy/dx}\right)\right]^2\Bigg|_{x=q_{1-\alpha}(\tilde{x})} \\
&= \frac{1}{6(1-\alpha)}\,\frac{1}{d\mu_{1,c}/dx}\,\frac{d}{dx}\left(\frac{\eta_{3,c}f_x}{d\mu_{1,c}/dx}\right) \\
&\quad + \frac{1}{8(1-\alpha)}\,\frac{1}{f_x}\,\frac{1}{d\mu_{1,c}/dx}\left[\frac{d}{dx}\left(\frac{\eta_{2,c}f_x}{d\mu_{1,c}/dx}\right)\right]^2\Bigg|_{x=q_{1-\alpha}(\tilde{x})},
\end{aligned}
\tag{4.78}
$$

which is our result for the ES-based second-order granularity adjustment in general form. As mentioned before, this adjustment formula is of order $O(1/n^2)$ because both the conditional skewness and the squared conditional variance are of this order.

4.3.2.4 Second-Order Granularity Adjustment for the Vasicek Model

As in Sect. 4.3.2.2 for the first-order adjustment, we now specify the second-order adjustment for the Vasicek model. Thus, we use the conditional PD of the Vasicek model

$$
p_i(x) = \Phi\left(\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i}\cdot x}{\sqrt{1-\rho_i}}\right)
\tag{4.79}
$$

and assume that the systematic factor is normally distributed. Due to the latter assumption, the second-order granularity adjustment (4.78) can be expressed as

$$\begin{aligned}\Delta l_2 &= \frac{1}{6(1-\alpha)} \frac{1}{d\mu_{1,c}/dx} \frac{d}{dx}\left(\frac{\eta_{3,c}\varphi}{d\mu_{1,c}/dx}\right) \\ &\quad + \frac{1}{8(1-\alpha)} \frac{1}{\varphi} \frac{1}{d\mu_{1,c}/dx} \left[\frac{d}{dx}\left(\frac{\eta_{2,c}\varphi}{d\mu_{1,c}/dx}\right)\right]^2 \Bigg|_{x=\Phi^{-1}(1-\alpha)} \\ &=: \Delta l_{2,1} + \Delta l_{2,2}\big|_{x=\Phi^{-1}(1-\alpha)}. \end{aligned} \tag{4.80}$$

As presented in Appendix 4.5.15, this leads to a second-order adjustment of

$$\begin{aligned}\Delta l_2 = &\frac{1}{6(1-\alpha)} \frac{\varphi}{\left(d\mu_{1,c}/dx\right)^2} \left[\frac{d\eta_{3,c}}{dx} - \eta_{3,c}\left(x - \frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\right] \\ &+ \frac{1}{8(1-\alpha)} \frac{\varphi}{\left(d\mu_{1,c}/dx\right)^3} \left[\frac{d\eta_{2,c}}{dx} - \eta_{2,c}\left(x - \frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\right]^2 \Bigg|_{x=\Phi^{-1}(1-\alpha)}. \end{aligned} \tag{4.81}$$

The required expressions for the conditional moments and the corresponding derivatives have already been determined in Sect. 4.2.1.4. Thus, we only have to insert the terms (4.37)–(4.47) into (4.81), which can easily be calculated with standard computer applications.

4.3.3 Moment Matching Procedure for Stochastic LGDs

Subsequently, we will study the accuracy of the ASRF formula and of the granularity adjustment for the risk measure ES in order to compare the capability of measuring name concentrations in comparison with the VaR (cf. Sect. 4.2.2). However, before we perform the corresponding numerical analyses, we deal with the modeling of stochastic LGDs. Based on this, we can perform our numerical analyses of the ES-based formulas not only for constant LGDs[217] but also for stochastic LGDs. This will show to which degree the accuracy of the ASRF framework and of the granularity adjustments are affected by this additional source of uncertainty. In order to incorporate a realistic degree of uncertainty, the probability distribution of LGDs will not be chosen on an ad-hoc basis, but different density functions will be parameterized in a way that mean and standard deviation

[217]Even if the calculations were based on the portfolio gross loss and thus on an LGD of 100%, the results remain identically for every constant LGD as the numerator and the denominator of the analyzed expressions are affected to the same degree.

will agree with empirical data reported by Schuermann (2005). These density functions, which are typically mentioned in the literature for modeling LGDs, are a normal distribution, a log-normal distribution, a logit-distribution, and a beta-distribution. This moment-matching procedure will be performed for senior secured, senior unsecured, senior subordinated, subordinated, as well as junior subordinated loans. As a next step, the 25%-, 50%-, and 75%-quantiles will be calculated for each of the parameterized distributions. Finally, the distribution with the smallest averaged difference between the calculated and the empirical quantiles will be chosen for the numerical analyses using the parameter setting for senior unsecured loans.

A typical shape of a recovery-rate-distribution, which is the distribution of $1-\text{LGD}$, can be seen in Fig. 4.8. The presented recovery rates correspond to 2,023 defaulted corporate bonds and loans from Moody's Default Risk Service Database. Approximately 88% of these instruments were issued by corporations domiciled in the United States.[218] In the presented case, the distribution is right-skewed, which means that there are many defaults with rather low recovery rates and few defaults with high recovery rates. While in most cases the recovery rate is between 0 and 100%, it is not necessarily bounded between these values. The demonstrated recovery rates of more than 100% appear if the interest rate at the time of recovery is lower than the coupon rate.[219] As mentioned in Sect. 2.2.1,

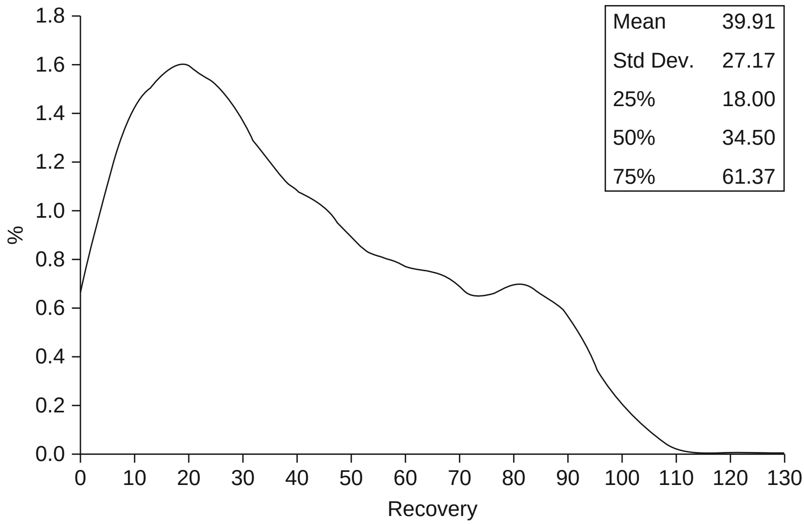

Fig. 4.8 Probability distribution of recovery rates for corporate bonds and loans, 1970–2003. See Schuermann (2005), p. 14

[218] Cf. Schuermann (2005), p. 22, footnote 8.

[219] Cf. Schuermann (2005), p. 22, footnote 11.

the case of recovery rates below 0% can occur due to workout costs. Since the attempt to recover a (part of a) loan is costly, the recovery rate is lower than 0% if the recovery cash flows are smaller than the workout costs. Even if this case is not presented in Fig. 4.8, it is practically more relevant than recovery rates of more than 100% as workout costs always occur whereas the other effect is if at all unsystematic.[220] Nonetheless, the mass of the distribution is between 0 and 100%, so that it can be beneficial to choose a probability distribution which is bounded between these values.

In the literature, there are different proposals for the choice of an LGD distribution. In the context of modeling LGDs that depend on a systematic factor,[221] Frye (2000) used the normal distribution. One point of criticism is that this distribution is symmetric and cannot describe the typically skewed LGDs. Against this background, Pykhtin (2003) chose the lognormal distribution. Schönbucher (2003) applied the logit-normal distribution, which is bounded between 0 and 1. As mentioned above, LGDs do not necessarily fulfill this characteristic but the distribution can almost be seen as bounded in this interval. A further common LGD distribution that is bounded in this interval is the beta distribution,[222] which is for example used in CreditMetrics™.[223] All of these distributions depend on two parameters. Thus, we can parameterize all of these distributions by matching the first two moments with the empirical distribution.

The probability density function of a *normally distributed* random variable $\tilde{X}$ is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \tag{4.82}$$

with mean μ and standard deviation σ, that is $\tilde{X} \sim \mathcal{N}(\mu, \sigma^2)$. The quantiles q_α of a normal distribution with parameters μ and σ can be calculated as

$$\begin{aligned} \mathbb{P}(\tilde{X} \leq q_\alpha) &= \Phi\left(\frac{q_\alpha - \mu}{\sigma}\right) = \alpha \\ &\Leftrightarrow \frac{q_\alpha - \mu}{\sigma} = \Phi^{-1}(\alpha) \\ &\Leftrightarrow q_\alpha = \mu + \sigma \cdot \Phi^{-1}(\alpha). \end{aligned} \tag{4.83}$$

[220]Probably, the data used to generate the figure did not include workout costs and therefore underestimate the true economic loss. Furthermore, the choice of the discount rate influences the effect of negative LGDs: If the recovery cash flows are discounted by the contractual rate, as required by IFRS and as proposed by the Basel II framework, a complete recovery without workout costs leads to a recovery rate of 100%, which shows that negative LGDs are not relevant at all.

[221]The issue of interconnections between LGDs and PDs via a systematic factor is not in the scope of this analysis.

[222]Cf. Altman et al. (2005), p. 46.

[223]Cf. Gupton et al. (1997), p. 80.

If a random variable $\tilde{X}$ is normally distributed with $\tilde{X} \sim \mathcal{N}(\mu_X, \sigma_X^2)$, the transformation $\tilde{Y} = e^{\tilde{X}}$ leads to a *lognormally distributed* variable $\tilde{Y}$.[224] The density function is

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_X^2}y} \exp\left(-\frac{(\ln y - \mu_X)^2}{2\sigma_X^2}\right). \tag{4.84}$$

In order to parameterize the distribution, the parameters μ_X and σ_X have to be expressed as a function of the known mean μ and standard deviation σ. Using the well-known moments of a lognormal distribution[225]

$$\mu = \exp\left(\mu_X + \frac{1}{2}\sigma_X^2\right) \quad \text{and} \quad \sigma^2 = (\exp(\sigma_X^2) - 1) \cdot \exp(2\mu_X + \sigma_X^2), \tag{4.85}$$

we obtain

$$\begin{aligned}
\sigma^2 &= \left(\exp\left(\sigma_X^2\right) - 1\right) \cdot \exp\left(2\mu_X + \sigma_X^2\right) \\
\Leftrightarrow \sigma^2 &= \left(\exp\left(\sigma_X^2\right) - 1\right) \cdot \exp\left(\mu_X + \frac{1}{2}\sigma_X^2\right)^2 \\
\Leftrightarrow \sigma^2 &= \left(\exp\left(\sigma_X^2\right) - 1\right) \cdot \mu^2 \\
\Leftrightarrow \sigma_X^2 &= \ln\left(\frac{\sigma^2}{\mu^2} + 1\right)
\end{aligned} \tag{4.86}$$

and

$$\begin{aligned}
\mu &= \exp\left(\mu_X + \frac{1}{2}\sigma_X^2\right) \\
\Leftrightarrow \mu_X &= \ln\mu - \frac{1}{2}\sigma_X^2 \\
\Leftrightarrow \mu_X &= \ln\mu - \frac{1}{2}\ln\left(\frac{\sigma^2}{\mu^2} + 1\right).
\end{aligned} \tag{4.87}$$

As the logarithm of a lognormally distributed variable is normally distributed with mean μ_X and standard deviation σ_X, the cumulative distribution function $F(y)$ can be expressed in terms of the standard normal distribution:

$$F_Y(y) = \Phi\left(\frac{\ln y - \mu_X}{\sigma_X}\right). \tag{4.88}$$

---

[224] See also Sect. 2.3.

[225] Cf. Bronshtein et al. (2007), p. 760, (16.80).

Similar to (4.83), this leads to

$$\Phi\left(\frac{\ln q_\alpha - \mu_X}{\sigma_X}\right) = \alpha$$
$$\Leftrightarrow q_\alpha = \exp\left(\mu_X + \sigma_X \cdot \Phi^{-1}(\alpha)\right). \tag{4.89}$$

A *logit-normal distribution* results from a normally distributed variable $\tilde{X}$ with $\tilde{X} \sim \mathcal{N}(\mu_X, \sigma_X^2)$, which is transformed by the logit function $\tilde{Y} = e^{\tilde{X}}/(1+e^{\tilde{X}})$. The transformation assures that the transformed variable is bounded to [0, 1]. As shown in Appendix 4.5.16, the probability density function is given as

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{(\ln(1/y-1)+\mu_X)^2}{2\sigma_X^2}\right)\frac{1}{y(1-y)}. \tag{4.90}$$

Since an analytical determination of mean and standard deviation is not obvious, the parameterization will be done numerically. For this purpose, the moments will be computed for different μ_X/σ_X-combinations until the deviation of both parameters from the empirical data is less than 10^{-4}. The corresponding quantiles will be determined via numerical integration of (4.90).

The density of a *beta distribution* with shape parameters $\alpha, \beta > 0$ can be defined as

$$f_X(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \tag{4.91}$$

where the beta function $B(\alpha,\beta)$ is defined as

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt \tag{4.92}$$

or as

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{4.93}$$

using the gamma function $\Gamma(\cdot)$.[226] With mean and variance

$$\mu = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(1+\alpha+\beta)}, \tag{4.94}$$

[226]Cf. Schönbucher (2003), p. 147 f.

the beta distribution can be parameterized using the following shape parameters

$$\mu = \frac{\alpha}{\alpha + \beta},$$
$$\Leftrightarrow \beta = \frac{\alpha}{\mu} - \alpha, \tag{4.95}$$

and

$$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(1+\alpha+\beta)}$$
$$\Leftrightarrow \sigma^2 = \frac{\alpha^2(1/\mu - 1)}{(\alpha/\mu)^2(1+\alpha/\mu)}$$
$$\Leftrightarrow \sigma^2 = \frac{\mu^2(1-\mu)}{(\mu+\alpha)}$$
$$\Leftrightarrow \alpha = \frac{\mu^2(1-\mu)}{\sigma^2} - \mu. \tag{4.96}$$

Similar to the logit-normal distribution, the quantiles of the beta distribution will be determined via numerical integration of (4.91).

As mentioned above, the different distribution functions will be parameterized using the data for corporate bonds and loans reported by Schuermann (2005). These data contain information about the empirical mean and standard deviation as well as the 25%-, 50%-, 75%-quantiles, and the number of observations N of recovery rates for different seniorities (see Table 4.8).[227] As expected, the average recovery rate as well as the quantiles of the recovery rate distribution are mostly the higher, the more senior the debt instrument.

In Tables 4.9–4.12, the determined parameters, which lead to a matching of moments, of the four considered distributions are reported for each of the seniorities. Furthermore, the corresponding quantiles $\hat{q}$ that result for these distributions are reported in the respective tables. The root mean squared errors (RMSE) are

Table 4.8 Recovery rates by seniority, 1970–2003[a]

Seniority	Mean μ	Std. dev. σ	$q_{0.25}$ (%)	$q_{0.5}$ (%)	$q_{0.75}$ (%)	N
Senior secured	0.543	0.258	33.00	53.50	75.00	433
Senior unsecured	0.387	0.278	14.50	30.75	63.00	971
Senior subordinated	0.285	0.234	10.00	23.00	42.25	260
Subordinated	0.347	0.222	19.50	30.29	45.25	347
Junior subordinated	0.144	0.090	9.13	13.00	19.13	12

[a]See Schuermann (2005), p. 16

[227]The aggregated data correspond to Fig. 4.8.

Table 4.9 Results of the normal distribution

Seniority	μ	σ	$\hat{q}_{0.25}$ (%)	$\hat{q}_{0.5}$ (%)	$\hat{q}_{0.75}$ (%)	RMSE (%)
Senior secured	0.543	0.258	36.84	54.26	71.68	2.97
Senior unsecured	0.387	0.278	19.96	38.71	57.46	6.43
Senior subordinated	0.285	0.234	12.72	28.51	44.30	3.74
Subordinated	0.347	0.222	19.66	34.65	49.64	3.57
Junior subordinated	0.144	0.090	8.33	14.39	20.45	1.20
						Ø 3.58

Table 4.10 Results of the lognormal distribution

Seniority	μ_X	σ_X	$\hat{q}_{0.25}$ (%)	$\hat{q}_{0.5}$ (%)	$\hat{q}_{0.75}$ (%)	RMSE (%)
Senior secured	−0.713	0.452	36.13	49.00	66.45	5.86
Senior unsecured	−1.157	0.645	20.35	31.44	48.58	9.00
Senior subordinated	−1.513	0.718	13.58	22.03	35.76	4.32
Subordinated	−1.232	0.587	19.63	29.16	43.33	1.28
Junior subordinated	−2.103	0.574	8.29	12.20	17.97	0.95
						Ø 4.28

Table 4.11 Results of the logit-normal distribution

Seniority	μ_X	σ_X	$\hat{q}_{0.25}$ (%)	$\hat{q}_{0.5}$ (%)	$\hat{q}_{0.75}$ (%)	RMSE (%)
Senior secured	0.234	1.396	33.02	55.82	76.41	1.57
Senior unsecured	−0.686	1.679	13.98	33.51	60.99	1.99
Senior subordinated	−1.284	1.493	9.20	21.70	43.13	1.02
Subordinated	−0.819	1.224	16.20	30.61	50.17	3.43
Junior subordinated	−1.967	0.741	7.83	12.28	18.75	0.89
						Ø 1.78

Table 4.12 Results of the beta distribution

Seniority	α	β	$\hat{q}_{0.25}$ (%)	$\hat{q}_{0.5}$ (%)	$\hat{q}_{0.75}$ (%)	RMSE (%)
Senior secured	1.477	1.245	33.59	55.43	75.84	1.26
Senior unsecured	0.801	1.269	14.04	34.58	60.63	2.61
Senior subordinated	0.775	1.944	8.61	22.85	44.01	1.30
Subordinated	1.241	2.341	16.27	31.55	50.37	3.57
Junior Subordinated	2.050	12.193	7.55	12.72	19.50	0.95
						Ø 1.94

reported as a quality criterion of the accuracy of the estimated quantiles in comparison with the empirical quantiles:

$$\text{RMSE} = \sqrt{\frac{1}{3}\left[(\hat{q}_{0.25} - q_{0.25})^2 + (\hat{q}_{0.5} - q_{0.5})^2 + (\hat{q}_{0.75} - q_{0.75})^2\right]}. \tag{4.97}$$

Finally, the averaged RMSE is reported for every distribution in order to determine the most appropriate description of an LGD distribution.

As can be seen from the tables, the normal and the lognormal distribution cannot fit the empirical data very well. By contrast, both the parameterized logit-normal and the beta distribution lead to a good accuracy with respect to the considered quantiles. As the logit-normal distribution leads to the smallest averaged RMSE, this distribution will be used to analyze the accuracy of the ASRF solution and the granularity adjustments for stochastic LGDs. For this purpose, the moments and the determined parameter values for *senior unsecured* bonds and loans will be implemented.

4.3.4 Numerical Analysis of the ES-Based Granularity Adjustment

4.3.4.1 Impact on the Portfolio-Quantile

In Sect. 4.2.2, we have studied the accuracy of the ASRF formula and the granularity adjustment for the risk measure VaR. However, we do not know how good the ES-based measurement of portfolio name concentration risk performs in comparison to the VaR-based measurement. Thus, our preceding analyses will be performed for the coherent ES subsequently. Moreover, we test the impact of stochastic LGDs on the accuracy of our approximation formulas. We start with an analysis of:

(a) The numerically "exact" coarse grained solution (see (2.76))
(b) The fine grained ASRF solution (see (4.59))
(c) The ASRF solution with first-order adjustment (see (4.70) and (4.73))
(d) The ASRF solution with first- and second-order adjustments (see (4.78) and (4.81))

for a homogeneous portfolio consisting of 40 credits with $PD = 1\%, LGD = 100\%$, and $\rho = 20\%$. The resulting ES using the formulas for the "exact" solution (a) as well as approximations (b) to (d) is presented in Fig. 4.9 for confidence levels starting at 0.7. In Fig. 4.10, the results for high confidence levels from 0.994 on are shown.

As can be seen in the figures, the ASRF solution underestimates the risk because the idiosyncratic component is neglected. Especially for high confidence levels, the impact of this underestimation is very high. The first-order granularity adjustment seems to be a very good approximation for a broad range of confidence levels. If the figures corresponding to the ES are compared to those of the VaR (see Figs. 4.1 and 4.2), the adjustment formula using the ES seems to work even better than the formula using the VaR. Unfortunately, it seems that the second-order adjustment cannot improve the result. Even if the approximation for high confidence levels is very good, the accuracy for lower confidence levels is significantly lower than without this additional adjustment.

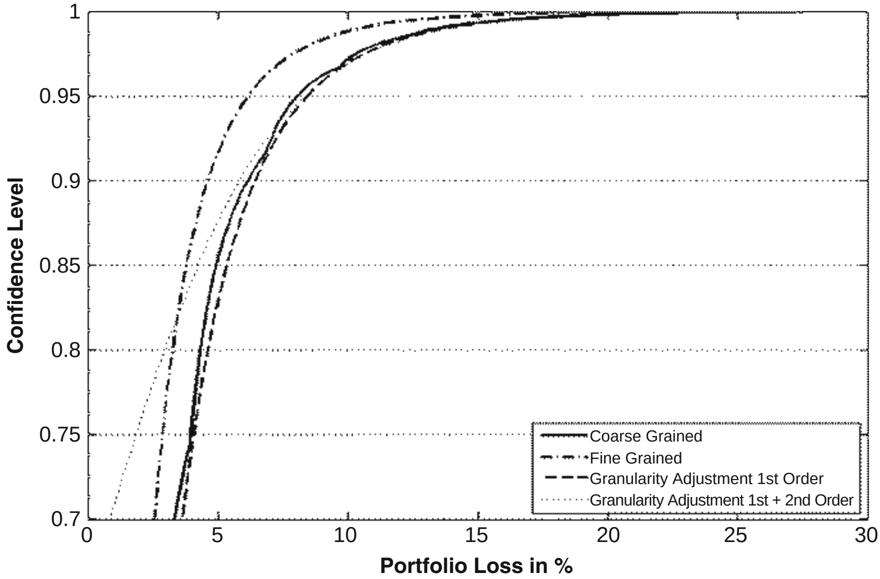

Fig. 4.9 Expected Shortfall for a wide range of probabilities

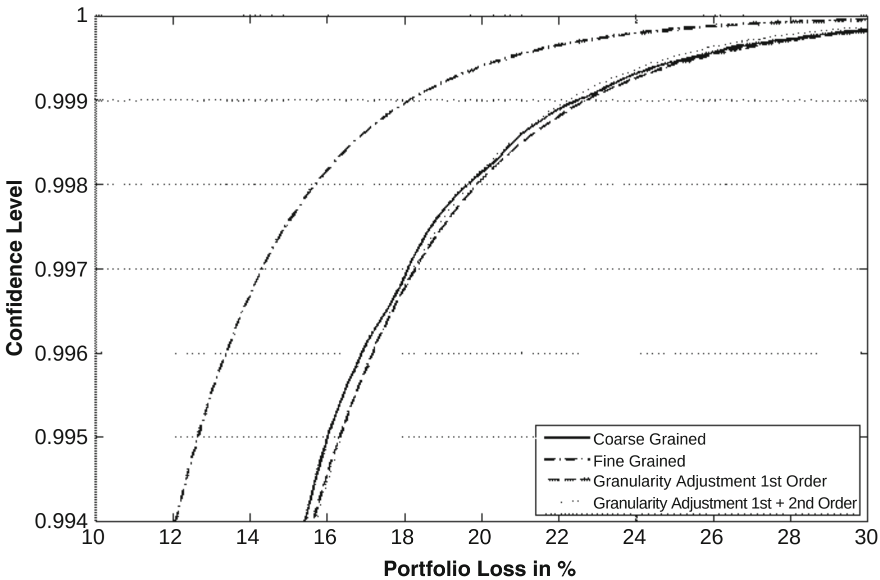

Fig. 4.10 Expected Shortfall for high confidence levels

In order to get a better insight in the accuracy of the different approximations, subsequently several numerical analyses will be performed similar to Sects. 4.2.2.2–4.2.2.4. In these sections, we have defined two kinds of critical numbers. The first measured the minimum number of credits a portfolio must consist of to have a good approximation of the "true" VaR at confidence level 0.999. The second number measured the critical number of credits for which the ASRF approximation of the 99.9%-VaR does not exceed the VaR at confidence level 0.995. Assuming that the increase of the confidence level from 0.995 to 0.999 happened to compensate the neglect of the granularity adjustment, it can be argued that the idiosyncratic risk component is already accounted for if the resulting critical number of credits is exceeded, whereas for a lower number of credits the risk is underestimated (for an actually intended confidence level of 0.995). The first type of analysis directly tests the performance of the different approaches. On the contrary, the second type of analysis does not focus on the accuracy of the approximation formulas but analyzes the need of additional economic capital against the specific regulatory setting. Thus, in order to test the performance of the different approximation formulas when using a different risk measure, only the first type of analyses will be performed in the following.[228] Due to the changed risk measure, the true risk will be given by the 99.72%-ES within the Vasicek model instead of the 99.9%-VaR.[229]

4.3.4.2 Size of Fine Grained Risk Buckets

Similar to Sect. 4.2.2.2, it will be determined for which portfolios the ES-based ASRF solution is a good approximation of the "true" ES. This will be done with a target tolerance of $\beta = 5\%$:[230]

$$I_{c,\text{ES,det.}}^{(\text{ASRF})} = \inf\left(n : \left| \frac{ES_{0.9972}^{(\text{ASRF})}\left(\tilde{L}\right)}{ES_{0.9972}^{(n)}\left(\tilde{L} = \frac{1}{n}\sum_{i=1}^{n} 1_{\{\tilde{D}_i\}}\right)} - 1 \right| < \beta \right) \text{ with } \beta = 0.05. \quad (4.98)$$

[228]The critical number of credits in a portfolio which leads to equality of the different parameter settings of the Basel consultative documents is not of interest in the subsequent analyses regarding the ES as both rely on the VaR.

[229]See Sect. 4.3.1.

[230]As the ASRF solution is constant and the coarse grained solution is monotonously decreasing in n for the ES (this is a result of the monotonicity of specific risk-property, cf. Sect. 4.3.1), the inequality also holds for every number above the first number that satisfies the inequality. Thus, the expression "for all $N \geq n$", which had to be included in the corresponding analysis for the VaR, can be neglected.

Table 4.13 Critical number of credits from that ASRF solution can be stated to be sufficient for measuring the true ES if LGDs are deterministic (see (4.98))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	30,405	20,112	4,711	4,516	3,593	2,803	1,828	1,246	893	443	346	191
3.5%	25,215	16,766	3,996	3,815	3,048	2,399	1,571	1,077	775	389	306	171
4.0%	21,425	14,273	3,460	3,297	2,644	2,079	1,375	946	686	348	275	155
4.5%	18,300	12,267	3,022	2,883	2,319	1,829	1,213	844	612	315	249	142
5.0%	15,920	10,714	2,663	2,561	2,054	1,628	1,090	758	553	286	228	132
5.5%	14,044	9,432	2,377	2,290	1,838	1,459	979	685	502	263	210	122
6.0%	12,434	8,443	2,140	2,058	1,658	1,319	889	625	461	243	195	113
6.5%	11,167	7,513	1,944	1,858	1,512	1,208	812	574	425	226	181	106
7.0%	9,985	6,786	1,765	1,701	1,374	1,100	750	529	393	211	170	101
7.5%	9,020	6,163	1,618	1,550	1,265	1,016	689	492	364	198	159	95
8.0%	8,201	5,617	1,490	1,426	1,169	933	641	456	342	186	150	90
8.5%	7,508	5,135	1,378	1,318	1,083	865	598	426	318	175	142	85
9.0%	6,922	4,709	1,277	1,222	1,007	805	555	400	299	166	135	81
9.5%	6,342	4,336	1,186	1,136	937	751	519	376	283	156	128	77
10.0%	5,833	4,054	1,104	1,059	874	702	487	354	267	149	122	74
10.5%	5,455	3,738	1,031	999	816	660	462	334	253	142	116	72
11.0%	5,035	3,462	974	933	764	623	434	315	240	136	111	68
11.5%	4,669	3,259	911	873	719	585	409	298	227	129	106	66
12.0%	4,386	3,021	854	824	681	551	386	283	216	123	102	64
12.5%	4,075	2,860	812	778	640	525	367	268	205	119	98	60
13.0%	3,845	2,657	762	732	611	495	349	257	196	114	94	58
13.5%	3,587	2,524	725	697	575	469	331	244	188	109	90	56
14.0%	3,389	2,351	684	657	545	447	318	233	179	105	87	54
14.5%	3,201	2,237	652	628	519	424	301	224	171	100	83	53
15.0%	3,002	2,095	617	593	493	405	290	213	166	97	80	51
15.5%	2,861	1,991	591	567	470	385	275	205	158	94	78	49
16.0%	2,684	1,905	558	538	452	369	265	196	152	90	75	47
16.5%	2,548	1,782	536	514	428	353	252	189	146	87	72	47
17.0%	2,438	1,703	508	495	411	337	244	181	141	85	71	45
17.5%	2,292	1,634	487	468	391	325	232	175	136	81	68	44
18.0%	2,181	1,532	469	450	375	309	224	167	131	79	66	42
18.5%	2,092	1,467	445	432	362	298	214	162	126	76	64	42
19.0%	1,998	1,411	428	411	344	288	207	155	123	74	62	40
19.5%	1,884	1,330	413	397	332	274	200	150	118	72	60	39
20.0%	1,806	1,273	393	384	321	265	191	146	115	69	59	37
20.5%	1,739	1,225	378	364	306	257	185	140	110	68	57	37
21.0%	1,653	1,182	366	351	295	244	180	136	107	65	56	37
21.5%	1,572	1,114	350	340	286	236	172	132	104	64	54	34
22.0%	1,512	1,070	336	324	273	229	167	126	100	62	53	34
22.5%	1,459	1,032	325	313	263	219	162	123	98	60	51	34
23.0%	1,411	999	315	303	255	212	155	120	94	59	50	32
23.5%	1,329	946	301	294	248	206	151	115	91	57	48	31
24.0%	1,277	908	290	280	237	200	146	112	89	56	47	31

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.); SMEs (Sales < 5 Mio.); Mortgage; Revolving retail; Other retail

Table 4.14 Critical number of credits from that ASRF solution can be stated to be sufficient for measuring the true ES if LGDs are stochastic (see (4.99))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	44,234	22,604	5,767	5,416	4,464	3,201	2,291	1,517	1,097	585	455	270
3.5%	28,168	20,206	4,362	4,764	3,597	2,615	1,785	1,312	1,022	476	397	245
4.0%	23,449	16,611	4,007	3,838	2,743	2,378	1,665	1,196	806	478	358	220
4.5%	21,337	16,066	3,438	3,592	2,877	2,393	1,423	1,039	855	403	316	207
5.0%	22,141	15,503	3,048	2,993	2,361	1,907	1,313	970	655	375	277	202
5.5%	20,044	11,914	2,600	3,112	2,197	1,497	1,157	794	623	351	265	172
6.0%	14,358	12,750	2,264	2,226	1,820	1,550	1,119	890	598	304	247	172
6.5%	17,261	10,528	2,174	2,283	1,852	1,461	909	637	550	325	248	159
7.0%	11,413	8,966	2,068	1,968	1,649	1,235	864	623	506	261	234	152
7.5%	10,555	10,372	1,718	1,728	1,481	1,379	851	627	506	237	210	149
8.0%	11,789	6,450	1,665	1,554	1,380	1,395	701	624	449	243	206	137
8.5%	11,395	6,049	1,605	1,672	1,307	1,086	651	463	391	227	206	129
9.0%	10,290	5,363	1,689	1,463	1,264	1,201	682	459	372	217	202	130
9.5%	6,833	6,043	1,588	1,432	1,028	853	737	474	373	203	171	121
10.0%	5,945	4,474	1,148	1,404	1,013	1,051	590	443	386	191	157	117
10.5%	8,491	3,458	1,197	1,283	1,012	818	594	462	346	180	157	113
11.0%	8,144	3,707	1,218	999	973	623	593	424	322	178	128	116
11.5%	4,860	3,684	1,066	1,103	752	864	405	376	282	180	145	106
12.0%	5,745	4,733	1,016	1,026	795	918	497	379	252	150	160	108
12.5%	5,918	3,352	1,032	903	756	677	502	315	253	156	133	107
13.0%	3,832	3,041	831	860	734	586	394	342	262	145	116	98
13.5%	4,284	2,810	1,005	884	805	558	397	310	292	149	127	95
14.0%	3,910	2,088	690	884	743	450	327	265	232	134	119	93
14.5%	4,854	3,034	876	683	741	495	428	245	215	132	119	91
15.0%	3,233	2,371	661	684	737	454	446	243	209	130	115	91
15.5%	3,357	3,308	858	551	583	529	323	314	163	126	97	90
16.0%	2,923	2,531	1,039	824	695	449	302	238	186	119	103	86
16.5%	4,623	1,675	630	609	643	416	433	214	182	117	106	84
17.0%	2,413	2,016	759	573	527	493	333	231	214	115	100	84
17.5%	2,406	2,145	517	468	430	384	280	235	190	122	92	82
18.0%	2,465	1,660	588	483	496	356	286	223	167	103	91	86
18.5%	3,963	2,814	600	476	543	436	222	197	144	99	89	80
19.0%	2,040	2,018	462	458	479	348	221	206	156	105	94	79
19.5%	2,533	1,331	421	500	488	320	246	216	154	97	88	76
20.0%	2,763	1,587	419	528	341	323	239	173	142	94	85	78
20.5%	2,408	1,490	535	505	476	354	230	205	163	98	80	77
21.0%	2,819	1,144	354	406	383	271	221	173	158	81	81	78
21.5%	2,106	1,105	380	503	372	227	202	172	125	114	87	75
22.0%	2,748	1,317	401	332	294	281	225	181	140	72	77	74
22.5%	2,709	1,185	450	311	370	249	169	149	127	81	76	71
23.0%	1,579	1,055	452	350	284	263	179	173	103	81	77	71
23.5%	1,785	2,476	384	430	269	258	181	132	148	80	72	71
24.0%	2,399	957	410	330	244	210	167	156	121	85	70	70

Corporates, sovereigns, and banks; SMEs (5Mio. $<$ Sales $<$ 50 Mio.); SMEs (Sales $<$ 5 Mio.); Mortgage; Revolving retail; Other retail

Moreover, we measure the accuracy of the ASRF solution if LGDs are stochastic and following a logit-normal distribution with

$$I_{c,\text{ES,stoch.}}^{(\text{ASRF})} = \inf\left(n : \left| \frac{ES_{0.9972}^{(\text{ASRF})}\left(\tilde{L}\right)}{ES_{0.9972}^{(n)}\left(\tilde{L} = \frac{1}{n}\sum_{i=1}^{n} \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}\right)} - 1 \right| < \beta \right) \text{with } \beta = 0.05. \tag{4.99}$$

In contrast to the analyses of Sect. 4.2.2 and the preceding definition of a critical number for deterministic LGDs (4.98), the denominator, which is the benchmark for the ASRF solution, cannot be determined with the Vasicek model because it does not account for stochastic LGDs. Against this background, we perform Monte Carlo simulations with one million trials for each PD/ρ-combination and for every number of credits until the target accuracy is reached.

The resulting critical numbers for the case of deterministic LGDs $I_{c,\text{ES,det.}}^{(\text{ASRF})}$ are reported in Table 4.13 for a broad range of correlations and PDs. Similar to the corresponding VaR-analysis, the values $I_{c,\text{ES,det.}}^{(\text{ASRF})}$ vary from 31 for a high PD/ρ-combination to 30,405 for a low PD/ρ-combination. This shows that at least for non-retail portfolios the assumption of infinite granularity is critical for real-world portfolios and the chosen risk measure does not influence the accuracy of the ASRF solution to a great extent.

The corresponding critical numbers for stochastic LGDs $I_{c,\text{ES,stoch.}}^{(\text{ASRF})}$ are reported in Table 4.14. As expected, the accuracy of the ASRF solution is lower for stochastic than for deterministic LGDs because there is an additional source of unsystematic uncertainty. In comparison with the case of deterministic LGDs, the minimum number of credits increased from a range between 31 and 31,405 to a range between 70 and 44,234 credits. On average, the required portfolio size is 31.55% higher due to stochastic LGDs if the identical accuracy shall be achieved.

4.3.4.3 Probing First-Order Granularity Adjustment

In order to test the accuracy of the ES-based first-order granularity adjustment, we determine the critical number $I_{c,\text{ES,det.}}^{(\text{1st Order Adj.})}$, which is the minimum number of credits to deliver a good approximation of the "true" ES on a 99.72% confidence level, for different PD/ρ-combinations. These critical values

$$I_{c,\text{ES,det.}}^{(\text{1st Order Adj.})} = \inf\left(n : \left| \frac{ES_{0.9972}^{(\text{1st Order Adj.})}\left(\tilde{L}\right)}{ES_{0.9972}^{(n)}\left(\tilde{L} = \frac{1}{N}\sum_{i=1}^{N} 1_{\{\tilde{D}_i\}}\right)} - 1 \right| < \beta \;\; \forall N \in \mathbb{N}^{\geq n} \right), \text{with } \beta = 0.05, \tag{4.100}$$

are presented in Table 4.15. As the ES-based first-order granularity adjustment does not only take the conditional variance of the default indicator into account but also the second moment of LGDs, it is interesting to find out how good the granularity adjustment performs in the presence of stochastic LGDs. For this purpose, we also determine the critical values

$$I_{c,\mathrm{ES,stoch.}}^{(1.\,\mathrm{Order\,Adj.})} = \inf\left(n : \left| \frac{ES_{0.9972}^{(1.\,\mathrm{Order\,Adj.})}\left(\tilde{L}\right)}{ES_{0.9972}^{(n)}\left(\tilde{L} = \frac{1}{N}\sum_{i=1}^{N} \widetilde{LGD}_i \cdot 1_{\left\{\tilde{D}_i\right\}}\right)} - 1 \right| < \beta \;\; \forall N \in \mathbb{N}^{\geq n} \right) \tag{4.101}$$

with $\beta = 0.05$, which are shown in Table 4.16.

For deterministic LGDs, the minimum number of credits varies between 7 and 2,468, which is a reduction of averaged 91.64% compared to the ASRF solution (see Table 4.13 in Sect. 4.3.4.2). Thus, we have a significant improvement of the accuracy if the first-order adjustment is taken into account. A very interesting finding results if the accuracy of the granularity adjustment is compared for the VaR and the ES. Even for a portfolio that consists of averaged 49.05% less credits and thus contains significantly more idiosyncratic risk, we are able to achieve the identical accuracy if name concentrations are measured on the basis of the Expected Shortfall instead of the Value at Risk. For the most relevant cases, where the minimum portfolio size is relatively high, this effect is even stronger.

If the improvement is analyzed only for cases where the minimum portfolio size is higher than 100 credits (determined for the VaR-based granularity adjustment), we find that the target accuracy can still be achieved if the portfolio consists of 68.91% less portfolios compared to a VaR-based measurement. For example, a high quality retail portfolio (AAA) must consist of at least 1,588 credits instead of 5,027 credits if name concentration is measured with the ES. Similarly, a medium quality corporate portfolio (BBB) must contain 25 compared to 106 credits. This shows that the already good performance of the VaR-based granularity adjustment can be improved significantly if name concentrations are measured with the ES.

The results for stochastic LGDs, which are presented in Table 4.16, are very promising. In most cases, the accuracy is slightly higher than in the case of deterministic LGDs. On average, the required portfolio size is reduced by 3.64%. Concretely, the accuracy is higher/identical/lower for 272/35/209 elements of the matrix. Of course, the results are influenced by a small degree of simulation noise but the accuracy seems to be at least identically in the presence of stochastic LGDs. If the accuracy of the granularity adjustment is compared with the ASRF solution of Table 4.14, the minimum number of credits is about 92.19% lower,[231] which is an excellent result. As a further robustness check, the corresponding values are determined for beta-distributed LGDs. In this case, the

[231] The corresponding value for deterministic LGDs is 91.64%.

Table 4.15 Critical number of credits from that the first order adjustment can be stated to be sufficient for measuring the true ES if LGDs are deterministic (see (4.100))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	2,468	1,870	439	466	367	279	192	148	111	62	53	38
3.5%	2,198	1,410	396	377	294	223	157	125	94	55	45	34
4.0%	1,588	1,010	313	298	266	205	145	106	81	48	40	29
4.5%	1,453	930	287	274	214	186	119	89	69	42	36	27
5.0%	976	858	224	213	198	152	111	83	64	37	34	25
5.5%	911	792	209	199	155	142	90	69	55	33	30	24
6.0%	853	726	195	186	146	112	85	65	52	31	27	22
6.5%	800	514	147	173	138	106	80	61	44	30	26	20
7.0%	752	485	139	133	129	100	64	51	42	27	23	20
7.5%	707	458	132	126	99	95	61	49	40	26	23	18
8.0%	665	433	126	120	94	89	58	47	33	25	22	18
8.5%	625	410	120	114	90	70	56	44	32	22	19	17
9.0%	585	250	113	108	86	67	53	36	31	21	19	16
9.5%	540	240	107	103	82	64	51	35	30	20	18	16
10.0%	358	231	101	74	79	62	40	34	29	20	16	13
10.5%	343	222	75	72	75	59	38	33	28	17	16	13
11.0%	330	213	72	69	71	57	37	25	23	17	16	13
11.5%	317	206	70	67	53	54	36	24	22	16	13	13
12.0%	305	198	67	64	51	52	35	24	22	16	13	13
12.5%	294	191	65	62	49	50	34	23	21	16	13	13
13.0%	283	185	63	60	48	37	33	23	20	13	13	11
13.5%	273	178	61	58	46	36	32	22	20	13	13	11
14.0%	264	172	59	56	45	35	31	21	19	13	12	11
14.5%	120	167	57	54	44	34	30	21	19	13	12	11
15.0%	117	161	55	53	42	33	29	20	18	12	12	11
15.5%	114	156	53	51	41	32	28	20	18	12	12	11
16.0%	111	151	51	33	40	31	26	19	14	12	12	11
16.5%	109	147	33	32	39	31	20	19	14	12	10	11
17.0%	106	142	33	31	37	30	20	18	14	12	10	11
17.5%	104	138	32	30	36	29	19	18	14	11	10	11
18.0%	101	134	31	30	35	28	19	18	13	11	10	11
18.5%	99	130	30	29	34	27	19	13	13	9	10	8
19.0%	97	63	30	28	23	27	18	13	13	9	9	9
19.5%	95	61	29	28	22	17	18	13	9	9	9	9
20.0%	93	60	28	27	22	17	17	12	9	9	9	9
20.5%	91	59	28	27	21	17	17	12	9	9	9	9
21.0%	89	58	27	26	21	16	17	12	9	9	9	9
21.5%	88	57	27	25	20	16	16	12	9	9	9	9
22.0%	86	56	26	25	20	16	16	11	9	9	7	9
22.5%	84	55	26	24	20	16	16	11	11	9	7	9
23.0%	83	54	25	24	19	15	15	11	11	8	7	9
23.5%	81	53	25	23	19	15	15	11	11	8	7	9
24.0%	80	52	24	23	19	15	15	11	11	8	7	9

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.); SMEs (Sales < 5 Mio.); Mortgage; Revolving retail; Other retail

Table 4.16 Critical number of credits from that the first order adjustment can be stated to be sufficient for measuring the true ES if LGDs are stochastic (see (4.101))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	2,338	1,682	531	470	403	308	243	158	126	74	66	50
3.5%	1,745	1,371	367	360	308	226	190	130	103	63	54	42
4.0%	1,663	1,104	315	308	241	214	151	117	89	54	49	39
4.5%	1,272	906	259	248	204	171	132	92	74	49	43	36
5.0%	1,055	779	225	224	175	164	112	89	64	41	37	33
5.5%	841	575	179	207	151	123	91	68	55	38	36	31
6.0%	758	506	165	158	140	107	85	70	53	35	33	29
6.5%	620	436	145	142	124	106	81	64	50	33	30	26
7.0%	595	416	126	122	129	94	63	63	46	30	27	26
7.5%	515	346	111	119	96	79	64	48	41	27	25	24
8.0%	473	335	101	107	89	72	61	42	36	25	25	23
8.5%	415	327	89	89	77	71	52	37	32	23	23	23
9.0%	272	290	79	86	75	66	48	38	32	23	22	21
9.5%	269	163	72	75	62	57	47	38	30	22	20	21
10.0%	233	170	74	69	64	56	36	32	27	20	19	19
10.5%	221	146	67	61	60	52	38	28	27	21	19	19
11.0%	189	146	64	60	58	50	34	35	25	20	18	19
11.5%	191	127	56	58	46	49	35	26	24	17	17	18
12.0%	174	119	56	54	45	35	35	23	23	18	16	17
12.5%	180	113	54	51	41	34	29	23	22	16	16	17
13.0%	169	111	51	48	37	31	30	22	22	15	14	16
13.5%	163	106	54	41	41	33	25	21	20	15	14	17
14.0%	142	102	42	41	35	33	23	22	19	15	14	15
14.5%	151	98	42	46	33	30	20	18	17	13	14	16
15.0%	139	92	42	44	30	28	25	18	16	12	13	16
15.5%	137	89	31	37	32	27	18	16	15	13	13	16
16.0%	133	89	45	36	31	27	19	16	15	12	13	15
16.5%	125	87	26	29	30	24	18	16	14	13	12	14
17.0%	131	79	36	24	20	23	17	16	13	11	12	14
17.5%	119	81	21	31	26	24	18	13	14	11	12	15
18.0%	105	81	21	23	25	22	15	12	13	11	11	15
18.5%	122	80	20	22	19	21	15	12	13	10	11	15
19.0%	109	77	21	19	16	17	15	11	12	11	10	14
19.5%	115	80	20	19	17	17	15	12	11	10	10	15
20.0%	112	69	18	18	15	17	15	11	11	10	10	14
20.5%	105	71	18	17	25	19	15	10	10	9	10	14
21.0%	102	69	17	15	14	16	14	10	10	10	9	14
21.5%	101	62	17	16	14	14	12	13	9	9	9	14
22.0%	92	62	17	15	13	14	14	10	8	8	9	13
22.5%	88	63	16	14	13	10	12	10	10	9	10	14
23.0%	86	67	15	14	12	14	11	10	9	9	9	14
23.5%	83	59	15	15	13	11	10	9	9	8	8	14
24.0%	97	58	14	15	12	10	12	9	9	8	8	14

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.)
SMEs (Sales < 5 Mio.); Mortgage; Revolving retail; Other retail

target accuracy is already reached for 4.89% less credits, compared to the case of deterministic LGDs. In comparison to the ASRF solution, the critical number is 92.27% lower.

4.3.4.4 Probing Second-Order Granularity Adjustment

As a next step, we analyze the accuracy of the ES-based second-order adjustment in comparison to the "exact" ES for deterministic LGDs:

$$I_{c,\mathrm{ES,det.}}^{(\mathrm{1st+2nd\,Order\,Adj.})} = \inf\left(n : \left| \frac{ES_{0.9972}^{(\mathrm{1st+2nd\,Order\,Adj.})}(\tilde{L})}{ES_{0.9972}^{(n)}\left(\tilde{L} = \frac{1}{N}\sum_{i=1}^{N} 1_{\{\tilde{D}_i\}}\right)} - 1 \right| < \beta \ \forall N \in \mathbb{N}^{\geq n} \right) \tag{4.102}$$

with $\beta = 0.05$. Moreover, the second order granularity adjustment is tested for stochastic LGDs using the formula

$$I_{c,\mathrm{ES,stoch.}}^{(1.+2.\,\mathrm{Order\,Adj.})} = \inf\left(n : \left| \frac{ES_{0.9972}^{(1.+2.\,\mathrm{Order\,Adj.})}(\tilde{L})}{ES_{0.9972}^{(n)}\left(\tilde{L} = \frac{1}{N}\sum_{i=1}^{N} \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}\right)} - 1 \right| < \beta \ \ \forall N \in \mathbb{N}^{\geq n} \right) \tag{4.103}$$

with $\beta = 0.05$. Due to the second-order adjustment, not only the variance but also the skewness of LGDs is accounted for in the approximation formula.

The results for deterministic LGDs, which are reported in Table 4.17, confirm the findings of Fig. 4.9 and also of the corresponding VaR-based analysis of Sect. 4.2.2.4. If concentration risk is measured with the second-order adjustment, the required portfolio size is 89.79% smaller than without the adjustment formula and it performs still better than the VaR-based adjustment formulas but there is no improvement compared to the ES-based first-order adjustment. Thus, it has to be stated that the second-order adjustment formula stemming from additional elements of the Taylor series expansion is performing worse than the first-order adjustment. As discussed in Sect. 4.2.2.4, it remains unclear if this unexpected result is e.g. a consequence of a non-converging Taylor series or if the consideration of more elements of the Taylor series could improve the approximation. But for all that, we found that the ES-based first-order adjustment is an excellent method for measuring name concentrations.

The corresponding results for stochastic LGDs are reported in Table 4.18. Interestingly, the results for low PDs and high correlation parameters are very good, whereas for high PDs and low correlation parameters the results are worse

Table 4.17 Critical number of credits from that the first plus second order adjustment can be stated to be sufficient for measuring the true ES if LGDs are deterministic (see (4.102))

	AAA up to AA− 0.03%	A− up to A+ 0.05%	BBB+ 0.32%	BBB 0.34%	BBB− 0.46%	BB+ 0.64%	BB 1.15%	BB− 1.97%	B+ 3.19%	B 8.99%	B− 13.01%	CCC up to C 30.85%
3.0%	3,381	2,533	880	841	707	585	433	338	271	178	159	131
3.5%	2,036	1,627	663	634	542	454	347	270	222	151	135	114
4.0%	1,302	1,127	491	473	413	355	279	223	183	130	118	103
4.5%	760	741	389	374	333	289	226	185	156	115	105	94
5.0%	594	443	306	295	269	237	189	159	136	102	94	86
5.5%	256	238	237	229	215	194	160	138	120	91	85	80
6.0%	466	161	180	176	169	157	135	120	107	84	78	74
6.5%	473	273	159	153	152	129	123	105	95	75	72	70
7.0%	746	453	113	110	116	113	103	91	84	69	67	66
7.5%	722	447	101	98	87	89	86	80	75	64	63	63
8.0%	695	435	66	65	76	80	80	73	67	60	59	59
8.5%	668	421	58	56	69	61	65	64	63	56	55	57
9.0%	641	407	33	50	46	54	61	59	56	52	52	55
9.5%	614	392	27	27	41	50	50	56	53	50	50	53
10.0%	588	378	23	23	37	35	45	48	50	47	47	51
10.5%	563	363	39	36	34	31	42	45	44	45	45	49
11.0%	539	350	40	38	18	28	40	43	42	42	43	48
11.5%	515	336	41	38	16	26	31	36	38	41	42	47
12.0%	492	323	64	60	14	15	29	34	36	38	39	45
12.5%	469	310	63	59	27	13	27	33	34	37	38	44
13.0%	445	298	62	59	27	12	26	28	33	36	37	43
13.5%	420	286	61	58	27	11	18	26	29	34	35	42
14.0%	292	274	60	56	42	19	17	25	28	33	35	42
14.5%	282	262	58	55	42	19	16	24	27	32	34	40
15.0%	272	178	57	54	41	19	15	23	26	31	32	40
15.5%	263	173	56	53	41	19	14	22	25	30	31	39
16.0%	254	168	54	52	40	29	9	18	25	29	31	38
16.5%	245	162	53	33	39	29	8	17	21	28	30	38
17.0%	237	158	52	33	38	28	8	16	21	27	30	37
17.5%	229	153	51	48	38	28	7	16	20	27	28	37
18.0%	221	148	33	47	37	28	7	15	19	26	28	37
18.5%	213	144	48	46	36	27	7	15	19	26	27	36
19.0%	206	139	47	45	36	27	6	14	18	25	27	35
19.5%	198	135	46	44	35	26	6	14	18	25	26	35
20.0%	191	131	45	43	34	17	6	14	18	23	26	35
20.5%	183	127	44	42	33	17	3	10	17	23	26	34
21.0%	176	123	43	41	33	17	3	10	15	23	25	34
21.5%	91	62	42	40	32	17	3	9	15	22	25	34
22.0%	88	60	41	39	31	16	4	9	14	22	25	34
22.5%	86	58	40	39	31	16	4	9	14	22	25	34
23.0%	83	57	39	38	30	23	4	9	14	21	23	33
23.5%	81	56	38	37	30	23	4	8	13	21	23	33
24.0%	78	54	37	36	29	23	4	8	13	21	23	33

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.)

SMEs (Sales < 5 Mio.); Mortgage; Revolving retail; Other retail

Table 4.18 Critical number of credits from that the first plus second order adjustment can be stated to be sufficient for measuring the true ES if LGDs are stochastic (see (4.103))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	4,175	3,045	1,045	980	818	699	499	393	327	227	201	181
3.5%	2,745	2,102	835	761	674	546	435	323	272	190	167	154
4.0%	1,699	1,410	618	579	548	424	331	275	218	165	148	144
4.5%	1,090	951	477	462	419	361	282	230	194	140	135	131
5.0%	541	632	398	396	347	272	252	184	163	128	119	120
5.5%	264	347	311	287	277	256	197	170	144	113	110	110
6.0%	288	210	254	258	210	198	162	136	130	105	96	104
6.5%	600	136	203	193	178	164	142	124	113	96	89	98
7.0%	652	388	158	159	137	139	131	105	101	84	82	92
7.5%	670	358	126	115	126	116	112	102	89	81	81	87
8.0%	670	376	95	93	103	108	91	86	90	75	72	85
8.5%	613	408	73	75	81	84	89	85	81	70	69	80
9.0%	555	368	47	46	64	70	77	73	72	67	65	80
9.5%	575	316	37	36	55	59	63	65	63	62	62	76
10.0%	531	364	24	29	38	48	62	63	61	60	61	75
10.5%	550	321	11	12	31	41	55	60	53	54	57	71
11.0%	495	323	35	18	23	30	46	45	51	53	55	70
11.5%	431	276	47	46	11	24	40	46	45	52	53	69
12.0%	366	278	54	49	8	22	34	44	44	49	51	69
12.5%	428	295	55	51	15	18	32	41	41	46	49	65
13.0%	424	271	55	50	18	16	27	37	36	45	47	65
13.5%	367	264	63	46	37	7	26	37	38	42	47	63
14.0%	225	233	52	49	34	6	24	31	34	41	46	65
14.5%	333	227	53	61	35	10	22	29	31	44	42	62
15.0%	215	220	54	53	35	24	16	27	31	40	42	63
15.5%	204	193	56	49	36	21	17	26	30	37	41	60
16.0%	191	189	54	46	36	25	13	24	28	37	40	60
16.5%	185	153	49	47	37	23	12	22	27	35	40	61
17.0%	169	128	50	46	34	23	11	21	25	34	37	60
17.5%	153	140	45	45	35	25	10	20	24	35	37	59
18.0%	138	145	44	44	33	24	9	19	25	33	35	59
18.5%	152	120	42	45	35	24	8	19	23	33	35	57
19.0%	130	113	52	42	31	22	4	17	22	33	36	58
19.5%	132	108	43	39	32	24	4	15	22	30	35	58
20.0%	133	90	40	46	31	23	3	16	21	30	34	58
20.5%	120	86	35	37	35	24	5	15	20	29	33	59
21.0%	113	85	38	40	29	22	5	13	21	29	33	59
21.5%	110	76	43	36	27	22	5	12	19	29	33	58
22.0%	102	73	36	36	28	23	6	12	19	28	34	59
22.5%	93	74	36	31	27	20	5	12	17	28	33	58
23.0%	86	77	34	33	26	22	6	11	18	27	32	59
23.5%	13	67	32	30	28	22	6	11	18	28	32	58
24.0%	24	67	31	34	24	22	6	11	16	28	31	59

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.); SMEs (Sales < 5 Mio.); Mortgage; Revolving retail; Other retail

than for the case of deterministic LGDs. Even if the required portfolio size is still significantly smaller than with the ASRF solution (–81.50%), the accuracy is worse than for deterministic LGDs (+16.25%). This confirms the findings from before that the first-order adjustment is strictly preferable. The corresponding values for beta-distributed LGDs are almost identical (–81.50% and +16.38%).

4.3.4.5 Probing Granularity for Inhomogeneous Portfolios

Subsequently, the accuracy of the ES-based granularity adjustment will be tested for inhomogeneous portfolios, which consist of credits with different exposure weights and default probabilities. The high quality and low quality test portfolios are identical to those of Sect. 4.2.2.5. The analyzed portfolios consist of 40, 60, . . ., 400, 800, 1,600, and 4,000 loans and the Expected Shortfall is computed at a confidence level of 99.72% for a correlation parameter of $\rho = 20\%$. The resulting first- and second-order granularity add-on and the corresponding ES of a Monte Carlo simulation with three million trials are presented in Fig. 4.11.

The size and shape of the true and the approximated granularity add-ons are similar to those calculated for the VaR. Thus, we find that for the portfolio

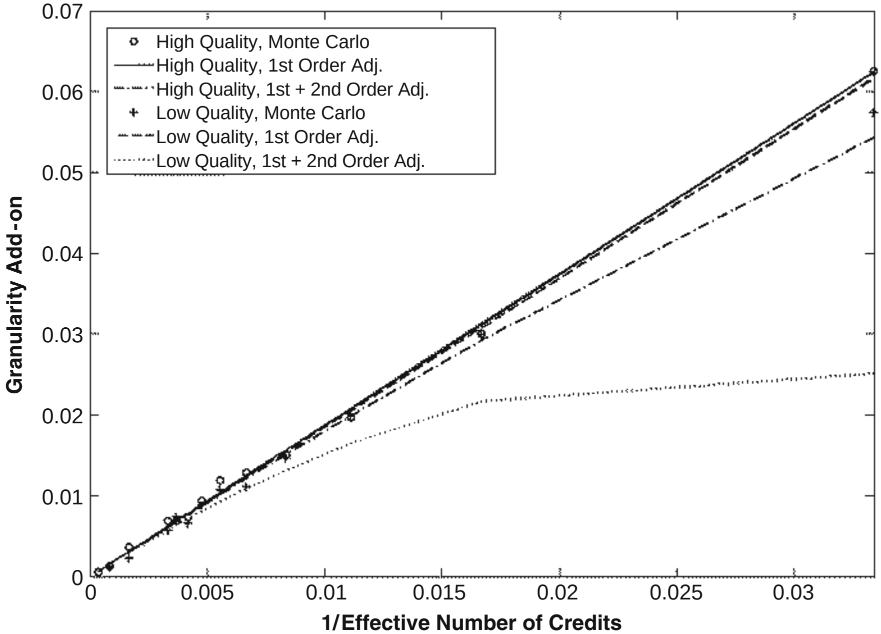

Fig. 4.11 ES-based granularity add-on for heterogeneous portfolios calculated analytically with first-order (*solid lines*) and second-order (*dotted lines*) adjustments as well as with Monte Carlo simulations (+ and o) using three million trials

consisting of 40 loans we have a granularity add-on of about 6%. In contrast to the VaR-based analysis, the add-on of the low-quality portfolio does not exceed the add-on of the high-quality portfolio. But most importantly, the granularity add-on is almost linear in terms of $1/n^*$ and the first-order adjustment is capable to capture the deviations from the ASRF solution with high accuracy, whereas the second-order adjustment leads to an underestimation of idiosyncratic risks.

4.4 Interim Result

Presently discussed analytical solutions for risk quantification of credit portfolio models especially rely on the assumptions of an infinite number of credits and of only one systematic factor. Thus, those analytical frameworks do not account for single name and sector concentration risk. This problem is discussed intensively by the financial authorities and it is especially considered in Pillar 2 of Basel II. To cope with the problem of name concentration, an add-on factor has been developed that adjusts the analytical solution for portfolios of finite size and therefore might serve as a simple solution for quantifying name concentration risk under Pillar 2. In this chapter, the general framework of this (first-order) granularity adjustment for medium sized risk buckets has been reviewed. Furthermore, we have derived an additional (second-order) adjustment for small risk buckets, which reduces the error term from $O(1/n^2)$ to $O(1/n^3)$. Even if it has already been mentioned by Gordy (2004) that it may be worthwhile to calculate these additional terms, the adjustment formula has not been determined before. After the derivation of the second-order-adjustment in general form, we have specified the formula for the Vasicek model. As a next step, we have carried out a detailed numerical study. In this study, we have reviewed the accuracy of the infinite granularity assumption for credit portfolios with a finite number of credits, as well as the improvement of accuracy with so-called first and second order granularity adjustments. Due to this study, banks are able to easily assess whether the assumption of infinite granularity is critical for their portfolio. Furthermore, the outcomes of the study show in which situations the granularity adjustment formulas are able to accurately measure portfolio name concentrations. These results are presented in terms of critical values for the minimum number of credits in a portfolio. We come to the conclusion that the critical number of credits for approving the assumption of infinite granularity is influenced by the probability of default, the asset correlation and of course the required accuracy of the analytical formula to great extent. We specify the minimum accuracy to 5%, i.e. if the credit portfolio is larger than our calculated critical values, the "true" risk and the approximation differ by less than 5%. This critical number of credits varies enormously, e.g. from 1,371 to 23,989 for a high-quality portfolio (A-rated) and from 23 to 205 for an extremely low-quality portfolio (CCC-rated) under the risk measure VaR. With the use of the first order granularity adjustment we can reduce these ranges drastically. The critical number of credits is in the bandwidth 456 to 4,227 (A-rated) and 9 to 42 (CCC-rated) and thus, the postulated accuracy should be obtained in many real-world portfolios. Additionally, the second order adjustment does not seem to work for the VaR since it reduces the add-on factor and the accuracy.

We have demonstrated that the VaR, which is coherent in the context of the ASRF framework, has some theoretical shortcomings if we leave the ASRF framework, which is necessary to account for name concentrations. For this reason, we have proposed a methodology how a more convenient risk measure can be used for the measurement of name concentrations. For this purpose, we have adjusted the confidence level of the ES in a way that the Pillar 1 formulas still lead to an almost identical capital requirement, leading to an ES-confidence level of $\alpha = 99.72\%$. Using this confidence level, we are able to measure name concentrations without being exposed to the theoretical shortcomings of the VaR, but the results are still consistent with the Pillar 1 formulas. Based on these preliminary considerations, we have theoretically derived the ES-based first- and second-order granularity adjustment in a general one-factor framework and for the Vasicek model. Similar to the corresponding formulas for the VaR, the second-order granularity adjustment, which is intended to improve the accuracy for small portfolios, has not been derived before in the literature. The subsequent numerical analyses confirm that the first-order granularity adjustment leads to a very good approximation of the unsystematic risk component whereas the second-order adjustment cannot improve the accuracy. Interestingly, the required portfolio size is not only 91.64% lower compared to the ASRF solution but also 49.05% lower compared to the VaR-based granularity adjustment. This shows that it is indeed advisable to measure name concentration risk on the basis of the coherent ES instead of relying on the non-coherent VaR.

These findings have been emphasized by a robustness check using stochastic LGDs. For this additional analysis, we have firstly calibrated several probability distributions with empirical data of recovery rates for different seniorities using a moment matching approach. Namely, we have used the normal distribution, the lognormal distribution, the logit-normal distribution, and the beta distribution. As the logit-normal distribution has performed best with respect to the empirical observed quantiles, we generated recovery rates which are logit-normal distributed with parameters stemming from the empirical data of senior unsecured loans. Using these data, we have repeated the test of the ASRF solution and the ES-based granularity adjustments. As expected, we find that the accuracy of the ASRF solution is lower due to the additional source of uncertainty. If the LGDs are stochastic, the minimum number of credits has to be 31.55% higher than for deterministic LGDs. Interestingly, the ES-based first-order adjustment performs slightly better in comparison with deterministic LGDs (4.89% less credits). Compared to the ASRF solution, the required portfolio size is 92.27% lower when using the first-order adjustment, which confirms our findings. Thus, apparently the accuracy of the measured risk is generally very high even for relatively small portfolios if the first-order granularity adjustment is incorporated.

4.5 Appendix

4.5.1 Alternative Derivation of the First-Order Granularity Adjustment

With reference to Wilde (2001), the granularity adjustment will be derived as an approximation of the difference Δq between the true VaR of a granular portfolio $q^{(n)}$ and the approximation $q^{(\infty)}$ that results if infinite granularity is assumed to hold:

$$\Delta q = q_\alpha^{(n)} - q_\alpha^{(\infty)}. \tag{4.104}$$

Instead of determining the add-on Δq directly, it will be analyzed how much the confidence level α will be overestimated or the probability $p := 1 - \alpha$ of exceeding the VaR will be underestimated if the portfolio is assumed to be infinitely granular. Thus, the probability

$$\Delta p = p^{(\infty)} - p = \alpha - \alpha^{(\infty)} \tag{4.105}$$

refers to the overestimation of the confidence level if only the systematic loss is considered. Here, α is the specified "target" confidence level, and by definition also the probability that the systematic loss will not exceed $q_\alpha^{(\infty)}$:

$$1 - p = \alpha := \mathbb{P}\left(\tilde{L} \le q_\alpha^{(n)}\right) = \mathbb{P}\left(\mathbb{E}\left[\tilde{L} \,|\, \tilde{x}\right] \le q_\alpha^{(\infty)}\right). \tag{4.106}$$

By contrast, $\alpha^{(\infty)}$ is the actual confidence level if the VaR is approximated by the ASRF model:

$$1 - p^{(\infty)} = \alpha^{(\infty)} := \mathbb{P}\left(\tilde{L} \le q_\alpha^{(\infty)}\right). \tag{4.107}$$

Subsequent to the derivation of Δp, the result will be transformed into a shift of the loss quantile Δq.

Analogous to Appendix 2.8.3, the unconditional probability $p^{(\infty)}$ can be expressed in terms of the conditional probability. Then, the substitution $y := q_\alpha^{(\infty)} + t$ is performed to center the integration at $q_\alpha^{(\infty)}$:

$$\begin{aligned} p + \Delta p = \mathbb{P}\left(\tilde{L} \ge q_\alpha^{(\infty)}\right) &= \int_{y=-\infty}^{\infty} \mathbb{P}\left(\tilde{L} \ge q_\alpha^{(\infty)} \,|\, \tilde{Y} = y\right) f_Y(y) dy \\ &= \int_{t=-\infty}^{\infty} \mathbb{P}\left(\tilde{L} \ge q_\alpha^{(\infty)} \,|\, \tilde{Y} = q_\alpha^{(\infty)} + t\right) f_Y\left(q_\alpha^{(\infty)} + t\right) dt, \end{aligned} \tag{4.108}$$

with the shorter notation $\tilde{Y} := \mathbb{E}(\tilde{L} \,|\, \tilde{x})$ for the conditional expectation. According to (4.106), the probability p can be written as

$$p = \mathbb{P}\left(\tilde{Y} \geq q_\alpha^{(\infty)}\right) = \int\limits_{y=q_\alpha^{(\infty)}}^{\infty} f_Y(y)\, dy = \int\limits_{t=0}^{\infty} f_Y\left(q_\alpha^{(\infty)} + t\right) dt \tag{4.109}$$

using the substitution $y := q_\alpha^{(\infty)} + t$ again, so that $t(y = q_\alpha^{(\infty)}) = 0$ and $t(y = \infty) = \infty$. Hence, (4.108) can be expressed as

$$\begin{aligned}
\Delta p = & \int\limits_{t=-\infty}^{\infty} \mathbb{P}\left(\tilde{L} \geq q_\alpha^{(\infty)} \,|\, \tilde{Y} = q_\alpha^{(\infty)} + t\right) f_Y\left(q_\alpha^{(\infty)} + t\right) dt - \int\limits_{t=0}^{\infty} f_Y\left(q_\alpha^{(\infty)} + t\right) dt \\
= & \int\limits_{t=-\infty}^{0} \mathbb{P}\left(\tilde{L} \geq q_\alpha^{(\infty)} \,|\, \tilde{Y} = q_\alpha^{(\infty)} + t\right) f_Y\left(q_\alpha^{(\infty)} + t\right) dt \\
& + \int\limits_{t=0}^{\infty} \left[\mathbb{P}\left(\tilde{L} \geq q_\alpha^{(\infty)} \,|\, \tilde{Y} = q_\alpha^{(\infty)} + t\right) - 1\right] f_Y\left(q_\alpha^{(\infty)} + t\right) dt.
\end{aligned} \tag{4.110}$$

The following transformations are performed for simplification of the integrand in order to solve the integral. A realization of the systematic loss implies a realization of the systematic factor. As the credit loss events are assumed to be independent for a realization of the systematic factor, the conditional credit losses follow a binomial distribution, which can be approximated by a normal distribution for a sufficient number of credits. This leads to

$$\begin{aligned}
\mathbb{P}\left(\tilde{L} \geq q_\alpha^{(\infty)} \,|\, \tilde{Y} = q_\alpha^{(\infty)} + t\right) &= 1 - \mathbb{P}\left(\tilde{L} < q_\alpha^{(\infty)} \,|\, \tilde{Y} = q_\alpha^{(\infty)} + t\right) \\
&\approx 1 - \Phi\left(\frac{q_\alpha^{(\infty)} - \mathbb{E}\left(\tilde{L} \,|\, \tilde{Y} = q_\alpha^{(\infty)} + t\right)}{\sqrt{\mathbb{V}\left(\tilde{L} \,|\, \tilde{Y} = q_\alpha^{(\infty)} + t\right)}}\right).
\end{aligned} \tag{4.111}$$

As $\mathbb{E}(\tilde{L}) = \mathbb{E}(\mathbb{E}(\tilde{L} \,|\, \tilde{x})) = \mathbb{E}(\tilde{Y})$, which is due to the law of iterated expectation, the conditional expectation of (4.111) equals

$$\mathbb{E}\left(\tilde{L} \,|\, \tilde{Y} = q_\alpha^{(\infty)} + t\right) = \mathbb{E}\left(\tilde{Y} \,|\, \tilde{Y} = q_\alpha^{(\infty)} + t\right) = q_\alpha^{(\infty)} + t. \tag{4.112}$$

With the symmetry $1 - \Phi(-x) = \Phi(x)$ and defining $\sigma^2(y) := \mathbb{V}(\tilde{L} \,|\, \tilde{Y} = y)$, (4.111) results in

$$\mathbb{P}\left(\tilde{L} \geq q_\alpha^{(\infty)} \mid \tilde{Y} = q_\alpha^{(\infty)} + t\right) \approx 1 - \Phi\left(\frac{q_\alpha^{(\infty)} - q_\alpha^{(\infty)} - t}{\sigma\left(q_\alpha^{(\infty)} + t\right)}\right)$$
$$= \Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)} + t\right)}\right), \tag{4.113}$$

so that (4.110) can be written as

$$\Delta p = \int_{t=-\infty}^{0} \Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)} + t\right)}\right) f_Y\left(q_\alpha^{(\infty)} + t\right) dt$$
$$+ \int_{t=0}^{\infty} \left[\Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)} + t\right)}\right) - 1\right] f_Y\left(q_\alpha^{(\infty)} + t\right) dt. \tag{4.114}$$

Subsequently, several linear approximations will be performed relying on the assumption that the loss quantile of the granular portfolio is close to the systematic loss quantile and the linearizations lead to minor errors. Linearizing the density function at $q_\alpha^{(\infty)}$ leads to

$$f_Y\left(q_\alpha^{(\infty)} + t\right) \approx f_Y\left(q_\alpha^{(\infty)}\right) + t \cdot \left.\frac{df_Y(y)}{dy}\right|_{y=q_\alpha^{(\infty)}}. \tag{4.115}$$

The argument of the normal distribution can be approximated as

$$t \cdot \left(\frac{1}{\sigma\left(q_\alpha^{(\infty)} + t\right)}\right) \approx t \cdot \left(\frac{1}{\sigma\left(q_\alpha^{(\infty)}\right)} + t \cdot \left[\frac{d}{dt}\frac{1}{\sigma\left(q_\alpha^{(\infty)} + t\right)}\right]_{t=0}\right)$$
$$= t \cdot \left(\frac{1}{\sigma\left(q_\alpha^{(\infty)}\right)} + t \cdot \left[-\frac{1}{\sigma^2\left(q_\alpha^{(\infty)} + t\right)}\frac{d}{dt}\sigma\left(q_\alpha^{(\infty)} + t\right)\right]_{t=0}\right)$$
$$= \left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)} - \frac{t^2}{\sigma^2\left(q_\alpha^{(\infty)}\right)}\left[\frac{d}{dt}\sigma\left(q_\alpha^{(\infty)} + t\right)\right]_{t=0}\right). \tag{4.116}$$

With the substitution $y := q_\alpha^{(\infty)} + t$, so $dy/dt = 1$ and $y(t=0) = q_\alpha^{(\infty)}$, the derivative of the conditional standard deviation can be rewritten as

$$\frac{d}{dt}\sigma\left(q_\alpha^{(\infty)} + t\right)\Big|_{t=0} = \frac{d}{dy}\sigma(y)\big|_{y=q_\alpha^{(\infty)}}. \tag{4.117}$$

Inserting (4.115)–(4.117) in (4.114) leads to

$$\begin{aligned}\Delta p =& \left(\int\limits_{t=-\infty}^{0} \Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)} - \frac{t^2}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \right)\right.\\ &\left.\cdot \left[f_Y\left(q_\alpha^{(\infty)}\right) + t \cdot \frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \right] dt \right)\\ &- \left(- \int\limits_{t=0}^{\infty} \left[\Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)} - \frac{t^2}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \right) - 1 \right]\right.\\ &\left.\cdot \left[f_Y\left(q_\alpha^{(\infty)}\right) + t \cdot \frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \right] dt \right)\\ &=: \Delta p_1 - \Delta p_2. \end{aligned} \tag{4.118}$$

When the substitution $t := -t$ for the term Δp_2 is performed and the symmetry of the normal distribution $\Phi(-x) - 1 = -\Phi(x)$ is used, both terms Δp_1 and Δp_2 are identical except for the algebraic signs:

$$\begin{aligned}\Delta p_2 =& - \int\limits_{t=0}^{-\infty} \left[\Phi\left(- \left[\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)} + \frac{t^2}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \right] \right) - 1 \right]\\ &\cdot \left[f_Y\left(q_\alpha^{(\infty)}\right) - t \cdot \frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \right] \cdot (-1) dt\\ =& \int\limits_{t=-\infty}^{0} \Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)} + \frac{t^2}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \right)\\ &\cdot \left(f_Y\left(q_\alpha^{(\infty)}\right) - t \cdot \frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \right) dt. \end{aligned} \tag{4.119}$$

A linearization of the normal distributions in Δp_1 and Δp_2 results in

$$\begin{aligned}&\Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)} \mp \frac{t^2}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \right)\\ &\approx \Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)} \right) \mp \frac{t^2}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \frac{d\Phi(y)}{dy}\bigg|_{y=\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)}}\\ &= \Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)} \right) \mp \frac{t^2}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \varphi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)} \right). \end{aligned} \tag{4.120}$$

Using this approximation, the terms Δp_1 and Δp_2 from (4.118) can be written as

$$\Delta p_{1,2} \approx \int\limits_{t=-\infty}^{0} \underbrace{\Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)}\right)}_{=:\beta_0} \cdot \left[\underbrace{f_Y\left(q_\alpha^{(\infty)}\right)}_{=:\gamma_0} \pm \underbrace{t\frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}}}_{=:\gamma_1}\right] dt$$
$$\mp \int\limits_{t=-\infty}^{0} \underbrace{\frac{t^2}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \varphi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)}\right)}_{=:\beta_1} \cdot \left[\underbrace{f_Y\left(q_\alpha^{(\infty)}\right)}_{=:\gamma_0} \pm \underbrace{t\frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}}}_{=:\gamma_1}\right] dt. \tag{4.121}$$

The summands β_0, γ_0 are the points around which the linearizations have been performed. The summands β_1, γ_1 have resulted from the first-order approximations. Using this notation, the shift in probability Δp of (4.118) can notably be simplified to

$$\begin{aligned} \Delta p &\approx \Delta p_1 - \Delta p_2 \\ &\approx \int\limits_{t=-\infty}^{0} \beta_0(\gamma_0+\gamma_1) - \beta_1(\gamma_0+\gamma_1)dt - \int\limits_{t=-\infty}^{0} \beta_0(\gamma_0-\gamma_1) + \beta_1(\gamma_0-\gamma_1)dt \\ &= \int\limits_{t=-\infty}^{0} 2\beta_0\gamma_1 - 2\beta_1\gamma_0 dt. \end{aligned} \tag{4.122}$$

Fortunately, both integrands are already first-order terms whereas the cross-terms $\beta_1 \cdot \gamma_1$ vanish.[232] Thus, there is no need for a further linearization. The remaining expression is

$$\begin{aligned} \Delta p \approx{}& 2\frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \int\limits_{t=-\infty}^{0} t \cdot \Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)}\right) dt \\ &- 2\frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \frac{f_Y\left(q_\alpha^{(\infty)}\right)}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \int\limits_{t=-\infty}^{0} t^2 \cdot \varphi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)}\right) dt. \end{aligned} \tag{4.123}$$

[232]The omission of the zeroth-order terms could be foreseen as only the *deviation* from the systematic loss quantile is analyzed.

In order to solve the integrals, the substitution $y := t/\sigma(q_\alpha^{(\infty)})$ is performed, with $dy/dt = 1/\sigma(q_\alpha^{(\infty)})$, $y(t=-\infty) = -\infty$ and $y(t=0) = 0$:

$$
\begin{aligned}
\Delta p &\approx 2\frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \int\limits_{y=-\infty}^{0} y\cdot\sigma\left(q_\alpha^{(\infty)}\right)\cdot\Phi(y)\cdot\sigma\left(q_\alpha^{(\infty)}\right)dy \\
&\quad - 2\frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \frac{f_Y\left(q_\alpha^{(\infty)}\right)}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \int\limits_{y=-\infty}^{0} \left[y\cdot\sigma\left(q_\alpha^{(\infty)}\right)\right]^2\cdot\varphi(y)\cdot\sigma\left(q_\alpha^{(\infty)}\right)dy \\
&= 2\frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \sigma^2\left(q_\alpha^{(\infty)}\right) \underbrace{\int\limits_{y=-\infty}^{0} y\cdot\Phi(y)dy}_{*} \\
&\quad - 2\frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} f_Y\left(q_\alpha^{(\infty)}\right)\cdot\sigma\left(q_\alpha^{(\infty)}\right) \underbrace{\int\limits_{y=-\infty}^{0} y^2\cdot\varphi(y)dy\,.}_{**}
\end{aligned}
\tag{4.124}
$$

For the second integral (**), it is used that the integrand is axially symmetric to the y-axis. Furthermore, the definition of the variance is utilized, considering that the standard normal distribution has mean $\mu_Y = 0$ and variance $\sigma_Y^2 = 1$:

$$
\begin{aligned}
\int\limits_{y=-\infty}^{0} y^2\cdot\varphi(y)dy &= \frac{1}{2}\int\limits_{y=-\infty}^{\infty} y^2\cdot\varphi(y)dy = \frac{1}{2}\int\limits_{y=-\infty}^{\infty} (y-\mu_Y)^2\cdot\varphi(y)dy. \\
&= \frac{1}{2}{\sigma_Y}^2 = \frac{1}{2}.
\end{aligned}
\tag{4.125}
$$

The first integral (*) can be calculated with integration by parts:

$$
\int\limits_{y=-\infty}^{0} y\cdot\Phi(y)dy = \left[\frac{1}{2}y^2\cdot\Phi(y)\right]_{y=-\infty}^{0} - \int\limits_{y=-\infty}^{0} \frac{1}{2}y^2\cdot\varphi(y)\,dy. \tag{4.126}
$$

For $y = 0$, the first term is zero but for $y = -\infty$, the result is not obvious. Using l'Hôpital's rule several times leads to[233]

[233]For functions f, g with $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = 0$ or $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = \infty$ it is true that $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)} = \lim\limits_{x\to x_0}\frac{f'(x)}{g'(x)}$ if $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)}$ exists; cf. Bronshtein et al. (2007), p. 54, (2.26).

$$
\begin{aligned}
\lim_{y\to-\infty}\frac{1}{2}y^2\cdot\Phi(y) &= \lim_{y\to\infty}\frac{1}{2}\frac{\Phi(-y)}{y^{-2}} \overset{\text{l'Hôpital}}{=} \lim_{y\to\infty}\frac{1}{2}\frac{-\varphi(-y)}{-2y^{-3}} \\
&= \lim_{y\to\infty}\frac{1}{4}\frac{y^3}{e^{y^2/2}} \overset{\text{l'Hôpital}}{=} \lim_{y\to\infty}\frac{1}{4}\frac{3y^2}{y\cdot e^{y^2/2}} \\
&= \lim_{y\to\infty}\frac{3}{4}\frac{y}{e^{y^2/2}} \overset{\text{l'Hôpital}}{=} \lim_{y\to\infty}\frac{3}{4}\frac{1}{y\cdot e^{y^2/2}} = 0,
\end{aligned}
\tag{4.127}
$$

so that the first term of (4.126) vanishes. Using the result of the previous integration, (4.126) equals $-1/4$. Hence, Δp from (4.124) is given as

$$
\Delta p \approx -\frac{1}{2}\frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}}\sigma^2\left(q_\alpha^{(\infty)}\right) - \frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} f_Y\left(q_\alpha^{(\infty)}\right)\cdot\sigma\left(q_\alpha^{(\infty)}\right). \tag{4.128}
$$

Because of $\sigma\frac{d\sigma}{dy} = \frac{1}{2}\frac{d\sigma^2}{d\sigma}\frac{d\sigma}{dy} = \frac{1}{2}\frac{d\sigma^2}{dy}$, (4.128) is equivalent to

$$
\begin{aligned}
\Delta p &\approx -\left[\frac{1}{2}\frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}}\sigma^2\left(q_\alpha^{(\infty)}\right) + \frac{1}{2}\frac{d\sigma^2(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} f_Y\left(q_\alpha^{(\infty)}\right)\right] \\
&= -\frac{1}{2}\left[\frac{df_Y(y)}{dy}\sigma^2(y) + \frac{d\sigma^2(y)}{dy}f_Y(y)\right]_{y=q_\alpha^{(\infty)}} \\
&= -\frac{1}{2}\frac{d}{dy}\left(f_Y(y)\cdot\sigma^2(y)\right)\big|_{y=q_\alpha^{(\infty)}}.
\end{aligned}
\tag{4.129}
$$

This expression is the linearized deviation of the specified probability $p = 1-\alpha$ if only the systematic loss is considered for calculation of the loss quantile.

As initially noticed, the determined shift of the probability has to be transformed into a shift of the loss quantile (cf. Fig. 4.12). If the probability density function of the portfolio loss is assumed to be almost linear in a region around the quantile, the required transformation is

$$
\Delta p \approx \frac{1}{2}\left[f_Y\left(q_\alpha^{(\infty)}\right) + f_Y\left(q_\alpha^{(\infty)} + \Delta q\right)\right]\Delta q. \tag{4.130}
$$

Two last first-order approximations lead to

$$
\begin{aligned}
\Delta p &\approx \frac{1}{2}\left[f_Y\left(q_\alpha^{(\infty)}\right) + \left(f_Y\left(q_\alpha^{(\infty)}\right) + \Delta q\frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}}\right)\right]\Delta q \\
&= f_Y\left(q_\alpha^{(\infty)}\right)\cdot\Delta q + \frac{1}{2}\frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}}(\Delta q)^2 \\
&\approx f_Y\left(q_\alpha^{(\infty)}\right)\cdot\Delta q.
\end{aligned}
\tag{4.131}
$$

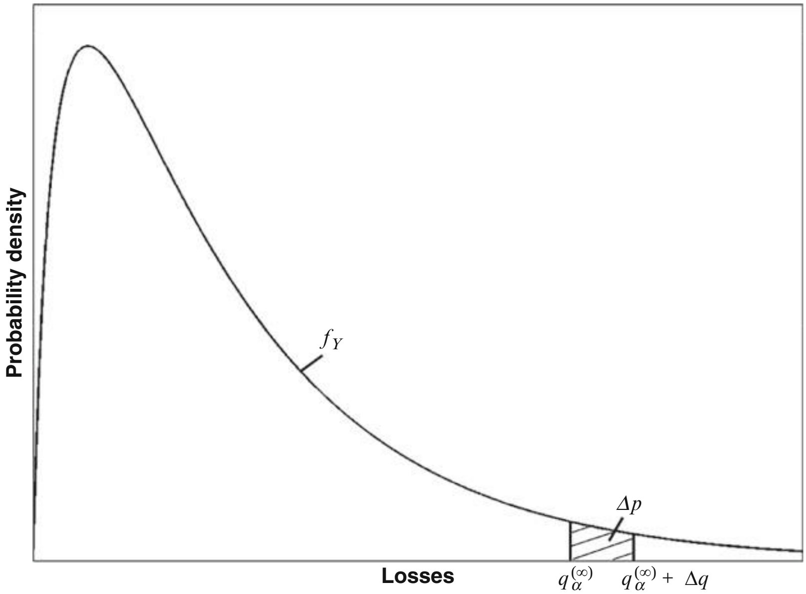

Fig. 4.12 Relation between the shift of the probability and the loss quantile

Inserting (4.129) into (4.131) finally leads to

$$\Delta q \approx \frac{\Delta p}{f_Y\left(q_\alpha^{(\infty)}\right)} \approx -\frac{1}{2}\frac{1}{f_Y(y)}\frac{d}{dy}\left(f_Y(y)\cdot\sigma^2(y)\right)\Big|_{y=q_\alpha^{(\infty)}}$$

$$= -\frac{1}{2}\frac{1}{f_Y(y)}\frac{d}{dy}\left(f_Y(y)\cdot\mathbb{V}\left(\tilde{L}\,|\,\tilde{Y}=y\right)\right)\Big|_{y=q_\alpha^{(\infty)}}. \tag{4.132}$$

Using (4.8), this can be written as

$$\Delta q \approx -\frac{1}{2f_x(x)}\frac{d}{dx}\left(\frac{f_x(x)\mathbb{V}\left[\tilde{L}\,|\,\tilde{x}=x\right]}{\frac{d}{dx}\mathbb{E}\left[\tilde{L}\,|\,\tilde{x}=x\right]}\right)\Bigg|_{x=q_{1-\alpha}(\tilde{x})}, \tag{4.133}$$

which is identical to the first-order granularity adjustment of Sect. 4.2.1.1.[234]

4.5.2 First and Second Derivative of VaR

The derivatives of VaR will be determined on the basis of Rau-Bredow (2002, 2004) in the following. Consider two continuous random variables $\tilde{Y}$ and $\tilde{Z}$ with

[234]Cf. Wilde (2001).

joint probability density function $f(y,z)$ and a variable $\lambda \in \mathbb{R}$. The VaR (the quantile) $q := q_\alpha(\tilde{L})$ of $\tilde{L} = \tilde{Y} + \lambda\tilde{Z}$ can implicitly be defined as[235]

$$\mathbb{P}(\tilde{L} \leq q) = \alpha. \tag{4.134}$$

Furthermore, the formula of the conditional density function will be used:[236]

$$f_{Z\,|\,Y=y}(z) = \frac{f_{Y,\,Z}(y,z)}{f_Y(y)}, \tag{4.135}$$

leading to[237]

$$f_{Z\,|\,Y+\lambda Z=q}(z) = \frac{f_{Y+\lambda Z,Z}(q,z)}{f_{Y+\lambda Z}(q)} = \frac{f_{Y,\,Z}(q-\lambda z,z)}{f_{Y+\lambda Z}(q)}. \tag{4.136}$$

4.5.2.1 First Derivative

As the derivative of the constant α is zero, the derivative of (4.134) is

$$\begin{aligned} 0 &= \frac{\partial}{\partial\lambda}\mathbb{P}(\tilde{Y} + \lambda\tilde{Z} \leq q) \\ &= \frac{\partial}{\partial\lambda}\int_{z=-\infty}^{\infty}\int_{y=-\infty}^{q-\lambda z} f_{Y,\,Z}(y,z)\,dy\,dz \\ &= \int_{z=-\infty}^{\infty}\frac{\partial}{\partial\lambda}\int_{y=-\infty}^{q-\lambda z} f_{Y,\,Z}(y,z)\,dy\,dz. \end{aligned} \tag{4.137}$$

Performing the inner integration and the differentiation leads to

$$0 = \int_{z=-\infty}^{\infty}\left(\frac{dq}{d\lambda} - z\right) f_{Y,\,Z}(q-\lambda z,z)\,dz. \tag{4.138}$$

[235]Cf. (2.14). The slightly different expressions compared to Rau-Bredow (2002) result from α instead of $(1–\alpha)$ representing the confidence level.

[236]Cf. Pitman (1999), p. 416.

[237]Cf. Rau-Bredow (2004), p. 66.

Using the formula for the conditional density function (4.135) and the integral representation of the conditional expectation, we get

$$\begin{aligned}
0 &= \int\limits_{z=-\infty}^{\infty} \left(\frac{dq}{d\lambda} - z\right) f_{Y+\lambda Z}(q) f_{Z\,|\,Y+\lambda Z=q}(z)\, dz \\
&= f_{Y+\lambda Z}(q)\left(\frac{dq}{d\lambda} \int\limits_{z=-\infty}^{\infty} f_{Z\,|\,Y+\lambda Z=q}(z)\, dz - \int\limits_{z=-\infty}^{\infty} z f_{Z\,|\,Y+\lambda Z=q}(z)\, dz\right) \\
&= f_{Y+\lambda Z}(q)\left(\frac{dq}{d\lambda}\cdot 1 - \mathbb{E}\big[\tilde{Z}\,|\,\tilde{Y} + \lambda\tilde{Z} = q\big]\right).
\end{aligned} \tag{4.139}$$

This leads to the first derivative of VaR:

$$\frac{dVaR_\alpha\big(\tilde{Y} + \lambda\tilde{Z}\big)}{d\lambda} = \mathbb{E}\big[\tilde{Z}\,|\,\tilde{Y} + \lambda\tilde{Z} = q_\alpha\big(\tilde{Y} + \lambda\tilde{Z}\big)\big]. \tag{4.140}$$

The first derivative at $\lambda = 0$ is

$$\left.\frac{dVaR_\alpha\big(\tilde{Y} + \lambda\tilde{Z}\big)}{d\lambda}\right|_{\lambda=0} = \mathbb{E}\big[\tilde{Z}\,|\,\tilde{Y} = q_\alpha\big(\tilde{Y}\big)\big]. \tag{4.141}$$

4.5.2.2 Second Derivative

Similar to (4.137), the second derivative of (4.134) is

$$0 = \frac{\partial^2}{\partial\lambda^2}\mathbb{P}\big(\tilde{Y} + \lambda\tilde{Z} \le q\big) = \frac{\partial^2}{\partial\lambda^2}\int\limits_{z=-\infty}^{\infty}\int\limits_{y=-\infty}^{q-\lambda z} f_{Y,\,Z}(y,z)\, dy\, dz. \tag{4.142}$$

With the first derivative of (4.138) and applying the product rule, this leads to

$$\begin{aligned}
0 &= \frac{\partial}{\partial\lambda}\int\limits_{z=-\infty}^{\infty}\left(\frac{dq}{d\lambda} - z\right) f_{Y,\,Z}(q-\lambda z, z)\, dz \\
&= \int\limits_{z=-\infty}^{\infty}\left(\frac{d^2 q}{d^2\lambda}\right) f_{Y,\,Z}(q-\lambda z, z) + \left(\frac{dq}{d\lambda} - z\right)\underbrace{\frac{\partial f_{Y,\,Z}(q-\lambda z, z)}{\partial\lambda}}_{*} dz.
\end{aligned} \tag{4.143}$$

The derivative (*) can be determined with the chain rule:

$$\begin{aligned}
\frac{\partial f_{Y,Z}(q-\lambda z,z)}{\partial\lambda} &= \frac{\partial(q-\lambda z)}{\partial\lambda}\,\frac{\partial f_{Y,Z}(q-\lambda z,z)}{\partial(q-\lambda z)}\,\frac{\partial q}{\partial q}\\
&= \left(\frac{dq}{d\lambda}-z\right)\frac{\partial f_{Y,Z}(q-\lambda z,z)}{\partial q}\,\frac{1}{\partial(q-\lambda z)/\partial q}\\
&= \left(\frac{dq}{d\lambda}-z\right)\frac{\partial f_{Y,Z}(q-\lambda z,z)}{\partial q}. \qquad (4.144)
\end{aligned}$$

Inserting (4.144) and the conditional density (4.136) into (4.143) results in

$$\begin{aligned}
0 &= \int\limits_{z=-\infty}^{\infty}\left(\frac{d^2q}{d^2\lambda}\right)f_{Y,Z}(q-\lambda z,z)+\left(\frac{dq}{d\lambda}-z\right)^2\frac{\partial f_{Y,Z}(q-\lambda z,z)}{\partial q}dz\\
&= \left(\frac{d^2q}{d^2\lambda}\right)\int\limits_{z=-\infty}^{\infty} f_{Y+\lambda Z}(q)\,f_{Z\,|\,Y+\lambda Z=q}(z)dz\\
&\quad+\int\limits_{z=-\infty}^{\infty}\left(\frac{dq}{d\lambda}-z\right)^2\frac{\partial\big(f_{Y+\lambda Z}(q)f_{Z\,|\,Y+\lambda Z=q}(z)\big)}{\partial q}dz. \qquad (4.145)
\end{aligned}$$

The first summand of (4.145) equals

$$\left(\frac{d^2q}{d^2\lambda}\right)f_{Y+\lambda Z}(q)\int\limits_{z=-\infty}^{\infty} f_{Z\,|\,Y+\lambda Z=q}(z)dz=\left(\frac{d^2q}{d^2\lambda}\right)f_{Y+\lambda Z}(q). \qquad (4.146)$$

In order to calculate the second summand of (4.145), the first derivative from (4.140) as well as the integral representation of the conditional variance is used:

$$\begin{aligned}
&\int\limits_{z=-\infty}^{\infty}\left(\frac{dq}{d\lambda}-z\right)^2\frac{\partial\big(f_{Y+\lambda Z}(q)f_{Z\,|\,Y+\lambda Z=q}(z)\big)}{\partial q}dz\\
&= \int\limits_{z=-\infty}^{\infty}\big(z-\mathbb{E}\big[\tilde{Z}\,|\,\tilde{Y}+\lambda\tilde{Z}=q\big]\big)^2\frac{\partial\big(f_{Y+\lambda Z}(q)f_{Z\,|\,Y+\lambda Z=q}(z)\big)}{\partial q}dz\\
&= \frac{d}{dy}\left(f_{Y+\lambda Z}(y)\int\limits_{z=-\infty}^{\infty}\big(z-\mathbb{E}\big[\tilde{Z}\,|\,\tilde{Y}+\lambda\tilde{Z}=q\big]\big)^2 f_{Z\,|\,Y+\lambda Z=y}(z)dz\right)\Bigg|_{y=q}\\
&= \frac{d}{dy}\big(f_{Y+\lambda Z}(y)\mathbb{V}\big[\tilde{Z}\,|\,\tilde{Y}+\lambda\tilde{Z}=y\big]\big)\Big|_{y=q}. \qquad (4.147)
\end{aligned}$$

With these summands, (4.145) can be written as

$$0 = \left(\frac{d^2 q}{d^2 \lambda}\right) f_{Y+\lambda Z}(y) + \frac{d}{dy}\left(f_{Y+\lambda Z}(y)\mathbb{V}\left[\tilde{Z} \mid \tilde{Y} + \lambda\tilde{Z} = y\right]\right)\bigg|_{y=q}. \tag{4.148}$$

Thus, the second derivative of VaR is equal to

$$\frac{d^2 VaR_\alpha\left(\tilde{Y} + \lambda\tilde{Z}\right)}{d^2 \lambda} = -\frac{1}{f_{Y+\lambda Z}(y)} \cdot \frac{d}{dy}\left(f_{Y+\lambda Z}(y)\mathbb{V}\left[\tilde{Z} \mid \tilde{Y} + \lambda\tilde{Z} = y\right]\right)\bigg|_{y=q_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}. \tag{4.149}$$

The second derivative at $\lambda = 0$ is

$$\frac{d^2 VaR_\alpha\left(\tilde{Y} + \lambda\tilde{Z}\right)}{d^2 \lambda}\bigg|_{\lambda=0} = -\frac{1}{f_Y(y)}\frac{d}{dy}\left(f_Y(y)\mathbb{V}\left[\tilde{Z} \mid \tilde{Y} = y\right]\right)\bigg|_{y=q_\alpha\left(\tilde{Y}\right)}. \tag{4.150}$$

4.5.3 Probability Density Function of Transformed Random Variables

Let $\tilde{X}$ be a random variable with density $f_X(x)$ and let $\tilde{Y}$ be a random variable with $\tilde{Y} = g(\tilde{X})$. If g is strictly monotonous and differentiable, the probability density function (PDF) of $\tilde{Y}$ can be transformed using the inverse function theorem[238]:

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \cdot \left|\frac{dg^{-1}(y)}{dy}\right|. \tag{4.151}$$

With $g^{-1}(y) = x$, we obtain

$$\left|\frac{dg^{-1}(y)}{dy}\right| = \left|\frac{dx}{dy}\right| = \left|\frac{1}{dy/dx}\right|, \tag{4.152}$$

which leads to

$$f_Y(y) = \frac{f_X(x)}{|dy/dx|}. \tag{4.153}$$

[238]Cf. Roussas (2007), p. 236.

4.5.4 VaR-Based First-Order Granularity Adjustment for a Normally Distributed Systematic Factor

The granularity adjustment (4.10) can be expressed as

$$\begin{aligned}
\Delta l_1 &= -\frac{1}{2\varphi}\frac{d}{dx}\left(\frac{\varphi\,\eta_{2,c}}{d\mu_{1,c}/dx}\right)\Bigg|_{x=\Phi^{-1}(1-\alpha)} \\
&= -\frac{1}{2\varphi}\left[\frac{d}{dx}\left(\varphi\,\eta_{2,c}\right)\frac{1}{d\mu_{1,c}/dx} + \varphi\,\eta_{2,c}\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right)\right]\Bigg|_{x=\Phi^{-1}(1-\alpha)} \\
&= -\frac{1}{2}\left[\frac{1}{\varphi}\frac{d}{dx}\left(\varphi\,\eta_{2,c}\right)\frac{1}{d\mu_{1,c}/dx} + \eta_{2,c}\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right)\right]\Bigg|_{x=\Phi^{-1}(1-\alpha)} \\
&= -\frac{1}{2}\left[\left(\frac{\eta_{2,c}}{\varphi}\frac{d\varphi}{dx} + \frac{d\eta_{2,c}}{dx}\right)\frac{1}{d\mu_{1,c}/dx} - \eta_{2,c}\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right]\Bigg|_{x=\Phi^{-1}(1-\alpha)}. \qquad (4.154)
\end{aligned}$$

Because of

$$\frac{1}{\varphi}\frac{d\varphi}{dx} = \frac{d(\ln\varphi)}{dx} = \frac{d}{dx}\left(\ln\left[\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\right]\right) = \frac{d}{dx}\left(\ln\frac{1}{\sqrt{2\pi}} - \frac{x^2}{2}\right) = -x, \qquad (4.155)$$

the granularity adjustment (4.154) can be written as

$$\Delta l_1 = \frac{1}{2}\left[\frac{x\cdot\eta_{2,c}}{d\mu_{1,c}/dx} - \frac{d\eta_{2,c}/dx}{d\mu_{1,c}/dx} + \frac{\eta_{2,c}\cdot d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right]\Bigg|_{x=\Phi^{-1}(1-\alpha)}. \qquad (4.156)$$

For the calculation of (4.156), the conditional expectation and variance have to be determined. Assuming stochastically independent LGDs and with *ELGD* and *VLGD* for the expectation and the variance of the LGD, respectively, the required moments are given as[239]

$$\begin{aligned}
\mu_{1,c} &= \mathbb{E}\left(\sum_{i=1}^{n} w_i\cdot\widetilde{LGD}_i\cdot 1_{\{\tilde{D}_i\}}\,|\,\tilde{x}=x\right) \\
&= \sum_{i=1}^{n} w_i\cdot ELGD_i\cdot\mathbb{E}\left(1_{\{\tilde{D}_i\}}\,|\,\tilde{x}=x\right) \\
&= \sum_{i=1}^{n} w_i\cdot ELGD_i\cdot p_i(x), \qquad (4.157)
\end{aligned}$$

[239]Pykhtin and Dev (2002) corrected the formulas of Wilde (2001), who neglected the last term of the following conditional variance.

$$
\begin{aligned}
\eta_{2,c} &= \mathbb{V}\left(\sum_{i=1}^{n} w_i \cdot \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} \,|\, \tilde{x} = x\right) \\
&= \sum_{i=1}^{n} w_i^2 \cdot \mathbb{V}\Big(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} \,|\, \tilde{x} = x\Big) \\
&= \sum_{i=1}^{n} w_i^2 \cdot \left[\mathbb{E}\left(\left[\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} \,|\, \tilde{x} = x\right]^2\right) - \mathbb{E}^2\Big(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} \,|\, \tilde{x} = x\Big)\right] \\
&= \sum_{i=1}^{n} w_i^2 \cdot \left[\mathbb{E}\Big(\widetilde{LGD}_i^{\,2}\Big) \cdot \mathbb{E}\left(\left[1_{\{\tilde{D}_i\}} \,|\, \tilde{x} = x\right]^2\right) - (ELGD_i \cdot p_i(x))^2\right] \\
&= \sum_{i=1}^{n} w_i^2 \cdot \left[\left(ELGD_i^2 + VLGD_i\right) \cdot p_i(x) - ELGD_i^2 \cdot p_i^2(x)\right].
\end{aligned}
\tag{4.158}
$$

4.5.5 VaR-Based First-Order Granularity Adjustment for Homogeneous Portfolios

For homogeneous portfolios, the granularity adjustment formula (4.28) can be simplified to

$$
\begin{aligned}
\Delta l_1 &= \frac{1}{2n}\Bigg[\Phi^{-1}(\alpha)\frac{(ELGD^2 + VLGD)\Phi(z) - ELGD^2\,\Phi^2(z)}{ELGD\left(\sqrt{\rho}/\sqrt{1-\rho}\right)\varphi(z)} \\
&\quad - \frac{(ELGD^2 + VLGD) - 2ELGD^2\,\Phi(z)}{ELGD} \\
&\quad - \frac{(ELGD^2 + VLGD)\Phi(z)z - ELGD^2\,\Phi^2(z)z}{ELGD \cdot \varphi(z)}\Bigg]_{z=\frac{\Phi^{-1}(PD)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}} \\
&= \frac{1}{2n}\Bigg(\frac{ELGD^2 + VLGD}{ELGD}\left[\frac{\sqrt{1-\rho}\,\Phi^{-1}(\alpha)\Phi(z)}{\sqrt{\rho}\,\varphi(z)} - 1 - \frac{\Phi(z)z}{\varphi(z)}\right] \\
&\quad - ELGD\,\Phi(z)\left[\frac{\sqrt{1-\rho}\,\Phi^{-1}(\alpha)\Phi(z)}{\sqrt{\rho}\,\varphi(z)} - 2 - \frac{\Phi(z)z}{\varphi(z)}\right]\Bigg)_{z=\frac{\Phi^{-1}(PD)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}} \\
&= \frac{1}{2n}\Bigg(\frac{ELGD^2 + VLGD}{ELGD}\left[\frac{\Phi(z)}{\varphi(z)}\,\frac{\Phi^{-1}(\alpha)(1-2\rho) + \Phi^{-1}(PD)\sqrt{\rho}}{\sqrt{\rho}\sqrt{1-\rho}} - 1\right] \\
&\quad - ELGD \cdot \Phi(z)\left[\frac{\Phi(z)}{\varphi(z)}\,\frac{\Phi^{-1}(\alpha)(1-2\rho) + \Phi^{-1}(PD)\sqrt{\rho}}{\sqrt{\rho}\sqrt{1-\rho}} - 2\right]\Bigg)_{z=\frac{\Phi^{-1}(PD)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}}.
\end{aligned}
\tag{4.159}
$$

4.5.6 Arbitrary Derivatives of VaR

The following determination of all derivatives of VaR is based on Wilde (2003). The quantile q_α of $\tilde{L} = \tilde{Y} + \lambda\tilde{Z}$ can be written as $q(\lambda)$ to denote that the quantile depends on the parameter λ. Using this notation, the quantile can be defined implicitly as an argument of the distribution function F by $F(q(\lambda), \lambda) := \mathbb{P}\big(\tilde{Y} + \lambda\tilde{Z} \leq q_\alpha(\tilde{Y} + \lambda\tilde{Z})\big) = \alpha$. In order to calculate the derivatives of q_α, at first all derivatives of F are determined in Sect. 4.5.6.2.1. As the quantile is defined implicitly, the implicit derivatives of $F(q(\lambda), \lambda) - \alpha = 0$ have to be determined. This is done by application of the residue theorem in Sect. 4.5.6.2.2. As a next step, the result will be expressed in combinatorial form in Sect. 4.5.6.2.3. Using the results of the derivatives of the distribution function and the implicit derivatives, it is possible to determine all derivatives of VaR. This is performed in Sect. 4.5.6.2.4. As the resulting formula is quite complex, an expression for the first five derivatives of VaR is determined in Sect. 4.5.7. The mathematical basics to the Laplace transform, complex residues, and partitions, which are needed within the derivation, are presented in the following Sect. 4.5.6.1.

4.5.6.1 Mathematical Basics

4.5.6.1.1 Laplace Transform and Dirac's Delta Function

The *Laplace transform* $\mathcal{L}$ of a function $f(t)$ with $t \in \mathbb{R}^+$ is given as[240]

$$[\mathcal{L}\{f(t)\}](s) := \int_{t=-0}^{\infty} f(t)e^{-st}dt =: \Theta(s) \tag{4.160}$$

with $s = c + i\omega \in \mathbb{C}$, where $\mathbb{C}$ denotes the set of all complex numbers. The *inverse Laplace transform* $\mathcal{L}^{-1}$ can be represented as[241]

$$\left[\mathcal{L}^{-1}\{\Theta(s)\}\right](t) := \frac{1}{2\pi i} \int_{s=c-i\infty}^{c+i\infty} \Theta(s)e^{st}ds = \mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\} = f(t). \tag{4.161}$$

Dirac's delta function $\delta(x)$ can be defined as[242]

$$\int_{-\infty}^{\infty} \delta(x)f(x - x_0)dx = f(x_0). \tag{4.162}$$

[240] Cf. Bronshtein et al. (2007), p. 710, (15.5).

[241] Cf. Bronshtein et al. (2007), p. 710, (15.8).

[242] Weisstein (2009a).

A more illustrative, heuristic definition of $\delta(x)$ is given by

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ \infty & \text{if } x = 0, \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)dx = 1. \tag{4.163}$$

Using the definition of the Laplace transform and the inverse Laplace transform, Dirac's delta function can be written as

$$\begin{aligned} \delta(t) &= \mathcal{L}^{-1}\{\mathcal{L}\{\delta(t)\}\} = \mathcal{L}^{-1}\left\{ \int_{t=-0}^{\infty} \delta(t)\mathrm{e}^{-st}dt \right\} \\ &= \mathcal{L}^{-1}\left\{\mathrm{e}^{-s\cdot 0}\right\} = \mathcal{L}^{-1}\{1\} = \frac{1}{2\pi i} \int_{s=c-i\infty}^{c+i\infty} 1 \cdot \mathrm{e}^{st}ds. \end{aligned} \tag{4.164}$$

4.5.6.1.2 Laurent Series, Singularities, and Complex Residues

If $f(z)$ is differentiable in all points of an open subset of the complex plane $H \subset \mathbb{C}$, then we call $f(z)$ *holomorphic* on H.[243] For a function $f(z)$, which is holomorphic in a simply connected region H, according to the *Cauchy integral theorem* we have[244]

$$\oint_C f(z)dz = 0, \tag{4.165}$$

with C being a closed path in H. If a function $f(z)$ is holomorphic in z_0 and in a circular region around z_0, we can perform a *Taylor series expansion*, which is analogous to the real plane:[245]

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n. \tag{4.166}$$

However, if a function $f(z)$ is only holomorphic inside the annulus between two concentric circles with center z_0 and radii r_1 and r_2, which is the region

[243] Cf. Bronshtein et al. (2007), p. 672, Sect. 14.1.2.1.

[244] Cf. Bronshtein et al. (2007), p. 688, (14.41).

[245] Cf. Bronshtein et al. (2007), p. 691, (14.49).

$H = \{z \,|\, 0 \le r_1 < |z - z_0| < r_2\}$, the function $f(z)$ can be expressed as a generalized power series, the so-called *Laurent series*:[246]

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \underbrace{\sum_{n=-\infty}^{-1} a_n (z - z_0)^n}_{\text{principal part}} + \underbrace{\sum_{n=0}^{\infty} a_n (z - z_0)^n}_{\text{analytic part}} . \tag{4.167}$$

Thus, the function has to be holomorphic only inside the annulus and not inside the inner circle or outside the outer circle.

If a function $f(z)$ is holomorphic in a neighborhood of z_0 but not in the point z_0, then z_0 is called an *isolated singularity* of the function $f(z)$. The concrete type of a singularity can be classified according to the analytic part of the Laurent series:[247]

- The point z_0 is a *removable singularity* if $a_n = 0 \;\forall n < 0$. In this case, the Laurent series is identical to the Taylor series above.
- The point z_0 is a *pole of order m* if the principal part consists of a finite number of terms with $a_m \neq 0$ and $a_n = 0$ for $n < m < 0$.
- The point z_0 is an *essential singularity* if the principal part consists of an infinite number of terms.

The coefficient a_{-1} of the Laurent series (4.167) around an isolated singularity z_0 is the *residue* of $f(z)$ in z_0. This will subsequently be denoted by $\mathrm{Res}_{z_0}(f)$. The residue can also be defined as

$$a_{-1} = \mathrm{Res}_{z_0}(f) = \frac{1}{2\pi i} \cdot \oint_C f(z)\, dz, \tag{4.168}$$

where C is a contour with winding number 1 in a holomorphic region H around an isolated singularity in z_0. If the contour C encloses a finite number of isolated singularities $z_1, z_2, ..., z_m$ with corresponding residues $a_{-1}(z_\mu)$ $(\mu = 1, ..., m)$, we have

$$\oint_C f(z)dz = 2\pi i \sum_{\mu=1}^{m} a_{-1}(z_\mu), \tag{4.169}$$

which is the *residue theorem*.[248]

The residue $\mathrm{Res}_{z_0}(f)$ with z_0 being a pole of order m can be calculated as[249]

$$\mathrm{Res}_{z_0}(f) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m \cdot f(z) \right]. \tag{4.170}$$

[246] Cf. Bronshtein et al. (2007), p. 692, (14.51), and Spiegel (1999), p. 144.
[247] Cf. Bronshtein et al. (2007), p. 692 f., Sect. 14.3.5.1.
[248] Cf. Bronshtein et al. (2007), p. 694, (14.56).
[249] Cf. Rowland and Weisstein (2009).

For a function $f = g(z)/h(z)$, where h has a simple zero in z_0, the residue can be determined with

$$\mathrm{Res}_{z_0}(f) = \frac{g(z_0)}{h'(z_0)}. \tag{4.171}$$

4.5.6.1.3 Partitions

A partition p of a positive integer m is a way to express m as a sum of positive integers in non-decreasing order. A partition p of m will be denoted by $p \prec m$. A partition p can be indicated by $p = 1^{e_1}, 2^{e_2}, \ldots, m^{e_m}$, where e_i is the frequency of the number i in the partition. The number of summands of p is expresses by $|p|$, which is the sum $|p| = e_1 + e_2 + \ldots + e_m$. The notation $\hat{p}$ indicates the partition which results if each summand of a partition p is increased by 1. This means that for $p \prec m$ the partition $\hat{p}$ refers to a specific partition of $m + |p|$.[250]

Example

- For $m = 5$, there exist seven partitions $p \prec m$: $p \prec m = \{1+1+1+1+1,\ 1+1+1+2,\ 1+2+2,\ 1+1+3,\ 2+3,\ 1+4,\ 5\}$. Thus, a concrete partition for $m = 5$ is $p = 3 + 1 + 1$.
- This partition can also be denoted by $p = 1^{e_1} 2^{e_2} \ldots m^{e_m} = 1^2 3^1$, leading to $e_1 = 2,\ e_2 = 0,\ e_3 = 1,\ e_4 = 0$, and $e_5 = 0$. Thus, the number m results from: $m = 1 \cdot e_1 + 2 \cdot e_2 + \ldots + m \cdot e_m = 1 \cdot 2 + 3 \cdot 1 = 5$.
- The number of summands of this partition is $\left|p = 1^2 3^1\right| = e_1 + e_2 + \ldots + e_m = 2 + 1 = 3$.
- The partition $\hat{p}$ appendant to the partition $p = 3 + 1 + 1$ is $\hat{p} = 4 + 2 + 2$, which is a specific partition of $m + |p| = 5 + 3 = 8$.

4.5.6.2 Determination of the Derivatives

4.5.6.2.1 Derivatives of the Distribution Function

Proposition. *The derivatives of the distribution function of losses $F_{Y+\lambda Z}(y) = \mathbb{P}(\tilde{Y} + \lambda\tilde{Z} \leq y)$ at $\lambda = 0$ are given as*[251]

$$\left.\frac{\partial^m}{\partial\lambda^m} F_{Y+\lambda Z}(y)\right|_{\lambda=0} = (-1)^m \frac{d^{m-1}}{dy^{m-1}} \left(\mathbb{E}\left(\tilde{Z}^m \mid \tilde{Y} = y\right) f_Y(y)\right). \tag{4.172}$$

---

[250] Cf. Wilde (2003), p. 3 f.

[251] See Martin and Wilde (2002), p. 124 f., and Wilde (2003), p. 2 f.

Proof. Using the definition of the Laplace transform (4.160) and recognizing that the loss $\tilde{L} = \tilde{Y} + \lambda\tilde{Z}$ cannot go below zero so that the probability density function is $f_{Y+\lambda Z}(y) = 0$ for all $y < 0$, we get for the Laplace transform of $f_{Y+\lambda Z}(y)$

$$\mathcal{L}\{f_{Y+\lambda Z}(y)\} = \int_{y=-0}^{\infty} \mathrm{e}^{-sy} f_{Y+\lambda Z}(y)dy = \int_{y=-\infty}^{\infty} \mathrm{e}^{-sy} f_{Y+\lambda Z}(y)dy. \tag{4.173}$$

With the definition of the expectation operator

$$\mathbb{E}\left(g\left(\tilde{X}\right)\right) = \int_{x=-\infty}^{\infty} g(x) f_X(x)dx, \tag{4.174}$$

(4.173) is equivalent to

$$\mathcal{L}\{f_{Y+\lambda Z}(y)\} = \int_{y=-\infty}^{\infty} \mathrm{e}^{-sy} f_{Y+\lambda Z}(y)dy = \mathbb{E}\left(\mathrm{e}^{-s\left(\tilde{Y}+\lambda\tilde{Z}\right)}\right). \tag{4.175}$$

Applying the definition of the inverse Laplace transform (4.161) and using the moment generating function M of $\tilde{Y} + \lambda\tilde{Z}$, which is defined as[252]

$$M_{Y+\lambda Z}(s) = \mathbb{E}\left(\mathrm{e}^{s\left(\tilde{Y}+\lambda\tilde{Z}\right)}\right), \tag{4.176}$$

the probability density function equals[253]

$$\begin{aligned} f_{Y+\lambda Z}(y) &= \mathcal{L}^{-1}\{\mathcal{L}\{f_{Y+\lambda Z}(y)\}\} = \mathcal{L}^{-1}\{M_{Y+\lambda Z}(-s)\} \\ &= \frac{1}{2\pi i} \int_{s=c-i\infty}^{c+i\infty} M_{Y+\lambda Z}(s)\mathrm{e}^{-sy}ds. \end{aligned} \tag{4.177}$$

Thus, the derivatives of the probability density function at $\lambda = 0$ can be determined using the approach

$$\left.\frac{\partial^m}{\partial\lambda^m} f_{Y+\lambda Z}(y)\right|_{\lambda=0} = \left.\frac{1}{2\pi i} \int_{s=c-i\infty}^{c+i\infty} \frac{\partial^m}{\partial\lambda^m} M_{Y+\lambda Z}(s)\mathrm{e}^{-sy}ds\right|_{\lambda=0}. \tag{4.178}$$

[252] Cf. Billingsley (1995), p. 146 ff., for details about moment generating functions.
[253] Cf. Miller and Childers (2004), p. 118.

Applying definition (4.176), we obtain for the derivatives of M

$$\begin{aligned}\left.\frac{\partial^m M_{Y+\lambda Z}(s)}{\partial\lambda^m}\right|_{\lambda=0} &= \left.\frac{\partial^m}{\partial\lambda^m}\mathbb{E}\left(\mathrm{e}^{s\left(\tilde{Y}+\lambda\tilde{Z}\right)}\right)\right|_{\lambda=0}\\ &= \left.\mathbb{E}\left(\frac{\partial^m}{\partial\lambda^m}\mathrm{e}^{s\left(\tilde{Y}+\lambda\tilde{Z}\right)}\right)\right|_{\lambda=0}\\ &= \left.\mathbb{E}\left(s^m\tilde{Z}^m\mathrm{e}^{s\left(\tilde{Y}+\lambda\tilde{Z}\right)}\right)\right|_{\lambda=0}\\ &= \mathbb{E}\left(s^m\tilde{Z}^m\mathrm{e}^{s\tilde{Y}}\right).\end{aligned} \tag{4.179}$$

With (4.179) and $s^m\mathrm{e}^{s\left(\tilde{Y}-y\right)} = (-1)^m\frac{\partial^m}{\partial y^m}\mathrm{e}^{s\left(\tilde{Y}-y\right)}$, (4.178) is equivalent to

$$\begin{aligned}\left.\frac{\partial^m}{\partial\lambda^m}f_{Y+\lambda Z}(y)\right|_{\lambda=0} &= \frac{1}{2\pi i}\int\limits_{s=c-i\infty}^{c+i\infty}\mathbb{E}\left(s^m\tilde{Z}^m\mathrm{e}^{s\tilde{Y}}\right)e^{-sy}ds\\ &= \mathbb{E}\left(\frac{1}{2\pi i}\tilde{Z}^m\int\limits_{s=c-i\infty}^{c+i\infty}s^m\mathrm{e}^{s\left(\tilde{Y}-y\right)}ds\right)\\ &= (-1)^m\frac{d^m}{dy^m}\mathbb{E}\left(\tilde{Z}^m\frac{1}{2\pi i}\int\limits_{s=c-i\infty}^{c+i\infty}\mathrm{e}^{s\left(\tilde{Y}-y\right)}ds\right).\end{aligned} \tag{4.180}$$

According to (4.164), Dirac's delta function can be written as

$$\delta(t) = \frac{1}{2\pi i}\int\limits_{s=c-i\infty}^{c+i\infty}1\cdot\mathrm{e}^{st}ds, \tag{4.181}$$

which leads to

$$\delta\left(\tilde{Y}-y\right) = \frac{1}{2\pi i}\int\limits_{s=c-i\infty}^{c+i\infty}1\cdot\mathrm{e}^{s\left(\tilde{Y}-y\right)}ds \tag{4.182}$$

for $t = \tilde{Y}-y$. Hence, (4.180) is equivalent to

$$\left.\frac{\partial^m}{\partial\lambda^m}f_{Y+\lambda Z}(y)\right|_{\lambda=0} = (-1)^m\frac{d^m}{dy^m}\mathbb{E}\left(\tilde{Z}^m\delta\left(\tilde{Y}-y\right)\right). \tag{4.183}$$

With $\mathbb{E}[\tilde{Z}^m \delta(\tilde{Y} - y)] = \mathbb{E}[\tilde{Z}^m \mid \tilde{Y} = y] \cdot f_Y(y)$, the derivatives of the distribution function result after integration of (4.183):

$$\frac{\partial^m}{\partial \lambda^m} F_{Y+\lambda Z}(y)\bigg|_{\lambda=0} = (-1)^m \frac{d^{m-1}}{dy^{m-1}} \left(\mathbb{E}\left(\tilde{Z}^m \mid \tilde{Y} = y\right) f_Y(y)\right), \tag{4.184}$$

which is proposition (4.172). In order to determine the derivatives of the quantile $d^m q/d\lambda^m$, the implicit derivatives of $F(q(\lambda),\lambda) - \alpha = 0$ with $F(q(\lambda),\lambda) := F_{\tilde{Y}+\lambda\tilde{Z}}(q_\alpha(\tilde{Y} + \lambda\tilde{Z})) = \mathbb{P}\left(\tilde{Y} + \lambda\tilde{Z} \le q_\alpha(\tilde{Y} + \lambda\tilde{Z})\right)$ will be calculated in the following.

4.5.6.2.2 Implicit Derivatives: Complex Residue Form

Consider a function $G(z,w)$ of two variables $z, w \in \mathbb{C}$. Suppose there exists an analytic function $w = w(z)$ in a region around a pole $z = z_0$, such that $G(z, w(z)) = 0$. The first derivative dw/dz can be determined as follows:[254]

$$\begin{aligned} 0 &= \frac{\partial G}{\partial z} + \frac{\partial G}{\partial w} \cdot \frac{dw}{dz} \\ \Leftrightarrow \frac{dw}{dz} &= -\frac{\partial G/\partial z}{\partial G/\partial w} =: -\frac{G_z}{G_w}. \end{aligned} \tag{4.185}$$

Proposition. *For $G_w(z_0, w_0) \neq 0$, the derivatives $d^m w/dz^m$ are given as*

$$\frac{d^m w}{dz^m} = -\mathrm{Res}_{w_0}\left[\frac{\partial^{m-1}}{\partial z^{m-1}}\left(\frac{G_z(z,w)}{G(z,w)}\right)\bigg|_{z=z_0}\right]. \tag{4.186}$$

Proof. According to (4.186), the first derivative is

$$\frac{dw}{dz} = -\mathrm{Res}_{w_0}\left[\left(\frac{G_z(z,w)}{G(z,w)}\right)\bigg|_{z=z_0}\right] = -\mathrm{Res}_{w_0}\left[\frac{G_z(z_0,w)}{G(z_0,w)}\right]. \tag{4.187}$$

As z_0 is a pole of G and $G(z_0, w) = 0$, an application of (4.171) leads to

$$\frac{dw}{dz} = -\mathrm{Res}_{w_0}\left[\frac{G_z(z_0,w)}{G(z_0,w)}\right] = -\frac{G_z}{G_w}, \tag{4.188}$$

[254] For ease of notation, the derivatives $\partial G/\partial z$ and $\partial G/\partial w$ will be abbreviated to G_z and G_w, respectively. The function G is not associated with a random variable, so confusion should not arise with respect to the similar notation $F_{Y+\lambda Z}(y)$, where the subscript of the distribution function F denotes the corresponding random variable.

which is equal to (4.185). This shows that the formula is correct for $m = 1$.

Applying the residue theorem (4.169)

$$\sum_{\mu=1}^{m} a_{-1}(z_\mu) = \frac{1}{2\pi i} \oint_C f(z)dz \tag{4.189}$$

and recognizing that there is only a singularity at $z = z_0$ leads to

$$\frac{dw}{dz} = -\mathrm{Res}_{w_0}\left[\left.\frac{G_z(z,w)}{G(z,w)}\right|_{z=z_0}\right] = -\frac{1}{2\pi i}\oint_C \left.\frac{G_z(z,w)}{G(z,w)}\right|_{z=z_0} dw. \tag{4.190}$$

Differentiating and applying the residue theorem again results in

$$\begin{aligned}\frac{d^m w}{dz^m} &= \frac{\partial^{m-1}}{\partial z^{m-1}}\left(-\frac{1}{2\pi i}\oint_C \left.\frac{G_z(z,w)}{G(z,w)}\right|_{z=z_0} dw\right)\\ &= -\frac{1}{2\pi i}\oint_C \left.\frac{\partial^{m-1}}{\partial z^m}\frac{G_z(z,w)}{G(z,w)}\right|_{z=z_0} dw\\ &= -\mathrm{Res}_{w_0}\left[\left.\frac{\partial^{m-1}}{\partial z^m}\frac{G_z(z,w)}{G(z,w)}\right|_{z=z_0}\right],\end{aligned} \tag{4.191}$$

which is the proposition presented in (4.186). This result is a generalization of the Lagrange inversion theorem.[255]

4.5.6.2.3 Implicit Derivatives: Combinatorial Form

In order to express the implicit derivatives (4.191) in combinatorial form, *Faà di Bruno's formula* will be used. According to this formula, the following equation holds for a function $g = g(y)$ with $y = y(x)$:[256]

$$\frac{d^m g}{dx^m} = \sum_{p \prec m} \alpha_p \frac{d^{|p|} g}{dy^{|p|}} \frac{d^p y}{dx^p}, \tag{4.192}$$

[255]Cf. Wilde (2003), p. 7.

[256]See Abramowitz and Stegun (1972), Sect. 24.1.2(C). The notation $p \prec m$ indicates that p is a partition of m, cf. Sect. 4.5.6.1.3.

with $\alpha_p = \frac{m!}{(1!)^{e_1}\cdot e_1!\cdot\ldots\cdot(m!)^{e_m}\cdot e_m!}$, $\frac{d^{|p|}g}{dy^{|p|}}$ as ordinary $|p|$th derivative, and

$$\frac{d^p y}{dx^p} := \left(\frac{dy}{dx}\right)^{e_{p1}} \cdot \left(\frac{d^2 y}{dx^2}\right)^{e_{p2}} \cdot \ldots \cdot \left(\frac{d^m y}{dx^m}\right)^{e_{pm}} = \prod_{i=1}^{m} \left(\frac{d^i y}{dx^i}\right)^{e_{pi}}. \tag{4.193}$$

Proposition. *Equation (4.191) is equivalent to*

$$\frac{d^m w}{dz^m} = \sum_{p \prec m, u \prec s \leq |p|-1} \alpha_p \alpha_{\hat{u}} \frac{(-1)^{|p|+|u|}(|p|+|u|-1)!}{(s+|u|)!(|p|-1-s)!} G_w{}^{-|p|-|u|} \frac{\partial^{\hat{u}} G}{\partial w^{\hat{u}}} \frac{\partial^{|p|-1-s}}{\partial w^{|p|-1-s}} \frac{\partial^p G}{\partial z^p}\bigg|_{z,w=0}. \tag{4.194}$$

Proof. For ease of notation, it will be assumed that $z_0 = w_0 = 0$, so that $G(0,0) = 0$. With $\partial \ln G/\partial z = G_z/G$, (4.191) is equivalent to

$$\frac{d^m w}{dz^m} = -\mathrm{Res}_{w_0}\left[\frac{\partial^{m-1}}{\partial z^{m-1}}\left(\frac{G_z}{G}\right)\bigg|_{z=0}\right] = -\mathrm{Res}_{w_0}\left[\frac{\partial^m}{\partial z^m} \ln G\bigg|_{z=0}\right]. \tag{4.195}$$

The mth derivative of $\ln G$ can be calculated using Faà di Bruno's formula:

$$\begin{aligned}\frac{\partial^m}{\partial z^m} \ln G &= \sum_{p \prec m} \alpha_p \frac{d^{|p|} \ln G}{dG^{|p|}} \frac{\partial^p G}{\partial z^p} = \sum_{p \prec m} \alpha_p \frac{d^{|p|-1}}{dG^{|p|-1}}\left(\frac{1}{G}\right)\frac{\partial^p G}{\partial z^p} \\ &= \sum_{p \prec m} \alpha_p \cdot (-1)^{|p|-1} \cdot (|p|-1)! \cdot G^{-|p|} \cdot G_{z,p},\end{aligned} \tag{4.196}$$

with $\partial^p G/\partial z^p =: G_{z,p}$. This leads to

$$\begin{aligned}\frac{d^m w}{dz^m} &= -\mathrm{Res}_{w_0}\left[\frac{\partial^m}{\partial z^m} \ln G\bigg|_{z=0}\right] \\ &= -\mathrm{Res}_{w_0}\left[\sum_{p \prec m} \alpha_p \cdot (-1)^{|p|-1} \cdot (|p|-1)! \cdot G^{-|p|} \cdot G_{z,p}\bigg|_{z=0}\right].\end{aligned} \tag{4.197}$$

According to (4.170), the residue of a function $h(w)$ in w_0, with w_0 being a pole of order r, can be calculated as

$$\mathrm{Res}_{w_0}[h(w)] = \lim_{w \to w_0} \frac{1}{(r-1)!} \frac{d^{r-1}}{dw^{r-1}}\left((w-w_0)^r \cdot h(w)\right). \tag{4.198}$$

With $r = |p|$, we obtain for the derivative (4.197)

$$\begin{aligned}
\frac{d^m w}{dz^m} &= -\mathrm{Res}_{w_0}\left[\sum_{p\prec m}\alpha_p\cdot(-1)^{|p|-1}\cdot(|p|-1)!\cdot G^{-|p|}\cdot G_{z,p}\Big|_{z=0}\right]\\
&= -\frac{1}{(|p|-1)!}\frac{\partial^{|p|-1}}{\partial w^{|p|-1}}\left[w^{|p|}\cdot\sum_{p\prec m}\alpha_p\cdot(-1)^{|p|-1}\cdot(|p|-1)!\cdot G^{-|p|}\cdot G_{z,p}\Big|_{z=0}\right]\Bigg|_{w=0}\\
&= -\sum_{p\prec m}\alpha_p\cdot(-1)^{|p|-1}\cdot\frac{\partial^{|p|-1}}{\partial w^{|p|-1}}\left(\left(\frac{G}{w}\right)^{-|p|}\cdot G_{z,p}\Big|_{z=0}\right)\Bigg|_{w=0}. \qquad (4.199)
\end{aligned}$$

Using the Leibniz identity for arbitrary-order derivatives of products of functions, we get:[257]

$$\begin{aligned}
\frac{d^m w}{dz^m} &= -\sum_{p\prec m}\alpha_p\cdot(-1)^{|p|-1}\cdot\frac{\partial^{|p|-1}}{\partial w^{|p|-1}}\left(\left(\frac{G}{w}\right)^{-|p|}\cdot G_{z,p}\Big|_{z=0}\right)\Bigg|_{w=0}\\
&= -\sum_{p\prec m}\alpha_p\cdot(-1)^{|p|-1}\cdot\sum_{s=0}^{|p|-1}\binom{|p|-1}{s}\\
&\quad\cdot\frac{\partial^s}{\partial w^s}\left(\frac{G(0,w)}{w}\right)^{-|p|}\cdot\frac{\partial^{|p|-1-s}}{\partial w^{|p|-1-s}}\left(G_{z,p}(0,w)\right)|_{w=0}. \qquad (4.200)
\end{aligned}$$

As a next step, the derivative $\frac{\partial^s}{\partial w^s}\left(\frac{G(0,w)}{w}\right)^{-|p|}$ contained in (4.200) will be calculated. Performing a Taylor series expansion of $G(0,w)$ at $w=0$, we have

$$\begin{aligned}
G(0,w) &= G(0,0)+\frac{w}{1!}\cdot\frac{\partial}{\partial w}G(0,0)+\frac{w^2}{2!}\cdot\frac{\partial^2}{\partial w^2}G(0,0)+\frac{w^3}{3!}\cdot\frac{\partial^3}{\partial w^3}G(0,0)+\ldots\\
&= 0+w\cdot G_w(0,0)+\sum_{r\geq 2}\frac{w^r}{r!}\cdot\frac{\partial^r}{\partial w^r}G(0,0)\\
&= w\cdot G_w(0,0)+\sum_{r\geq 1}\frac{w^{r+1}}{(r+1)!}\cdot\frac{\partial^{r+1}}{\partial w^{r+1}}G(0,0)\\
&= w\cdot G_w(0,0)+w\cdot G_w(0,0)\cdot\sum_{r\geq 1}\frac{w^r}{(r+1)!}\cdot\frac{\partial^{r+1}}{\partial w^{r+1}}G(0,0)\cdot\frac{1}{G_w(0,0)}\\
&= w\cdot G_w(0,0)\cdot\left(1+\sum_{r\geq 1}\frac{w^r}{r!}\cdot\frac{1}{r+1}\cdot\frac{\frac{\partial^{r+1}}{\partial w^{r+1}}G(0,0)}{\frac{\partial}{\partial w}G(0,0)}\right). \qquad (4.201)
\end{aligned}$$

[257]See Weisstein (2009b).

Thus, for $G(0,w)/w$, we obtain

$$\frac{G(0,w)}{w} = G_w(0,0) \cdot \left(1 + \sum_{r\geq 1} \frac{w^r}{r!} \cdot \frac{1}{r+1} \cdot \frac{\frac{\partial^{r+1}}{\partial w^{r+1}} G(0,0)}{\frac{\partial}{\partial w} G(0,0)}\right)$$
$$= G_w(0,0) \cdot \left(1 + \sum_{r\geq 1} \frac{w^r}{r!} \cdot \varphi_r\right), \tag{4.202}$$

with $\varphi_r = \frac{1}{r+1} \cdot \frac{\partial^{r+1}/\partial w^{r+1} G(0,0)}{\partial/\partial w G(0,0)}$. Another application of Faà di Bruno's formula results in:[258]

$$\frac{\partial^s}{\partial w^s}\left(\frac{G(0,w)}{w}\right)^{-|p|} = G_w^{-|p|}(0,0) \cdot \frac{\partial^s}{\partial w^s}\left(1 + \sum_{r\geq 1} \varphi_r \cdot \frac{w^r}{r!}\right)^{-|p|}$$
$$= G_w^{-|p|}(0,0) \cdot \sum_{u \prec s} \alpha_u \cdot \varphi_u \cdot (-1)^{|u|} \cdot \frac{(|p|+|u|-1)!}{(|p|-1)!}, \tag{4.203}$$

with[259]

$$\alpha_u \cdot \varphi_u = \frac{s!}{(s+|u|)!} \cdot \alpha_{\hat{u}} \cdot \frac{\partial^{\hat{u}}}{\partial w^{\hat{u}}} G(0,0) \cdot {G_w}^{-|u|}(0,0). \tag{4.204}$$

Applying (4.203) and (4.204) to (4.200) leads to

$$\frac{d^m w}{dz^m} = -\sum_{p \prec m} \alpha_p \cdot (-1)^{|p|-1} \cdot \sum_{s=0}^{|p|-1} \binom{|p|-1}{s} \frac{\partial^s}{\partial w^s}\left(\frac{G(0,w)}{w}\right)^{-|p|} \cdot \frac{\partial^{|p|-1-s}}{\partial w^{|p|-1-s}}\left(G_{z,p}(0,w)\right)\Big|_{w=0}$$
$$= -\sum_{p \prec m} \alpha_p \cdot (-1)^{|p|-1} \cdot \sum_{s=0}^{|p|-1} \binom{|p|-1}{s} \cdot G_w^{-|p|}(0,0) \cdot \sum_{u \prec s} \frac{s!}{(s+|u|)!} \cdot \alpha_{\hat{u}} \cdot \frac{\partial^{\hat{u}}}{\partial w^{\hat{u}}} G(0,0)$$
$$\cdot {G_w}^{-|u|}(0,0) \cdot (-1)^{|u|} \cdot \frac{(|p|+|u|-1)!}{(|p|-1)!} \cdot \frac{\partial^{|p|-1-s}}{\partial w^{|p|-1-s}}\left(G_{z,p}(0,w)\right)\Big|_{w=0}$$
$$= -\sum_{p \prec m} \alpha_p \cdot (-1)^{|p|-1} \cdot \sum_{s=0}^{|p|-1} \binom{|p|-1}{s} \cdot \sum_{u \prec s} \alpha_{\hat{u}} \cdot (-1)^{|u|} \cdot G_w^{-|p|-|u|}(0,0)$$
$$\cdot \frac{s! \cdot (|p|+|u|-1)!}{(s+|u|)! \cdot (|p|-1)!} \cdot \frac{\partial^{\hat{u}}}{\partial w^{\hat{u}}} G(0,0) \cdot \frac{\partial^{|p|-1-s}}{\partial w^{|p|-1-s}}\left(G_{z,p}(0,w)\right)\Big|_{w=0}. \tag{4.205}$$

[258]Cf. Wilde (2003), p. 8.

[259]The relation between a partition u and $\hat{u}$ is explained in Sect. 4.5.6.1.3.

Summarizing the sums, using $(-1)\cdot(-1)^{|p|-1}\cdot(-1)^{|u|}=(-1)^{|p|+|u|}$, and

$$\begin{aligned}
&\binom{|p|-1}{s}\cdot\frac{s!}{(|p|-1)!}\cdot\frac{(|p|+|u|-1)!}{(s+|u|)!}\\
&\quad=\frac{(|p|-1)!}{s!\cdot(|p|-1-s)!}\cdot\frac{s!}{(|p|-1)!}\cdot\frac{(|p|+|u|-1)!}{(s+|u|)!}\\
&\quad=\frac{(|p|+|u|-1)!}{(|p|-1-s)!\cdot(s+|u|)!},
\end{aligned}\tag{4.206}$$

(4.205) can be simplified to

$$\begin{aligned}
\frac{d^m w}{dz^m}=&\sum_{p\prec m,u\prec s\leq|p|-1}\alpha_p\cdot\alpha_{\hat{u}}\cdot(-1)^{|p|+|u|}\cdot\frac{(|p|+|u|-1)!}{(|p|-1-s)!\cdot(s+|u|)!}\\
&\cdot G_w^{-|p|-|u|}(0,0)\cdot\frac{\partial^{\hat{u}}}{\partial w^{\hat{u}}}G(0,0)\\
&\cdot\frac{\partial^{|p|-1-s}}{\partial w^{|p|-1-s}}\left(G_{z,p}(0,w)\right)\bigg|_{w=0},
\end{aligned}\tag{4.207}$$

which concludes the proof.

4.5.6.2.4 Completion of the Derivation

Application of (4.207) can be used to determine the derivatives of a quantile, which will be calculated subsequently. With $F(q(\lambda),\lambda)-\alpha=0=G(w(z),z)$, the derivatives are given as

$$\frac{d^m q}{d\lambda^m}\bigg|_{\lambda=0}=\frac{d^m w}{dz^m}\bigg|_{z=0},\tag{4.208}$$

where the right-hand side can be determined with (4.207). The derivatives of G contained in (4.207) can be calculated with (4.172):

$$\begin{aligned}
\frac{\partial^{r+s}G}{\partial w^r\partial z^s}\bigg|_{z=0}&=\frac{\partial^{r+s}F}{\partial y^r\partial\lambda^s}\bigg|_{\lambda=0}=\frac{\partial^r}{\partial y^r}\left(\frac{\partial^s F}{\partial\lambda^s}\right)\bigg|_{\lambda=0}\\
&=\frac{d^r}{dy^r}\left((-1)^s\frac{d^{s-1}}{dy^{s-1}}\left(\mathbb{E}\big(\tilde{Z}^s\mid\tilde{Y}=y\big)f_Y(y)\right)\right)\\
&=(-1)^s\frac{d^{r+s-1}}{dy^{r+s-1}}\left(\mathbb{E}\big(\tilde{Z}^s\mid\tilde{Y}=y\big)f_Y(y)\right)\\
&=(-1)^s\frac{d^{r+s-1}}{dy^{r+s-1}}\left(\mu_{s,c}f\right),
\end{aligned}\tag{4.209}$$

where we define $\mu_{s,c} := \mathbb{E}(\tilde{Z}^s \mid \tilde{Y} = y)$ and $f := f_Y(y)$ for convenience. Using definition (4.193) for the pth derivative with $p \prec m$, this leads to

$$\left.\frac{\partial^p G}{\partial z^p}\right|_{z=0} = \left.\prod_{i=1}^{m}\left(\frac{\partial^i G}{\partial z^i}\right)^{e_{pi}}\right|_{z=0} = \prod_{i=1}^{m}\left((-1)^i \frac{d^{i-1}(\mu_{i,c}f)}{dy^{i-1}}\right)^{e_{pi}} = (-1)^m \prod_{i=1}^{m}\left(\frac{d^{i-1}(\mu_{i,c}f)}{dy^{i-1}}\right)^{e_{pi}}. \tag{4.210}$$

Similarly the $\hat{u}$th derivative can be determined with $u \prec s$. It has to be considered that for each partition u the elements of the corresponding partition $\hat{u}$ are increased by 1. Thus, the smallest number is 2 and the largest is $s+1$. Hence, we obtain

$$\begin{aligned}\frac{\partial^{\hat{u}} G}{\partial w^{\hat{u}}} &= \prod_{i=2}^{s+1}\left(\frac{\partial^i G}{\partial w^i}\right)^{e_{\hat{u}i}} = \prod_{i=2}^{s+1}\left(\frac{\partial^i G}{\partial w^i}\right)^{e_{u(i-1)}} = \prod_{i=1}^{s}\left(\frac{\partial^{i+1} G}{\partial w^{i+1}}\right)^{e_{ui}} \\ &= \prod_{i=1}^{s}\left(\frac{\partial^{i+1} F}{\partial y^{i+1}}\right)^{e_{ui}} = \prod_{i=1}^{s}\left(\frac{d^i f}{dy^i}\right)^{e_{ui}}.\end{aligned} \tag{4.211}$$

Furthermore, we have $G_w = dF/dy = f$ and $(-1)^{|p|+|u|} \cdot f^{|p|+|u|} = (-f)^{|p|+|u|}$. Using these formulas, we finally get for (4.207) or (4.208):

$$\begin{aligned}\left.\frac{d^m q}{d\lambda^m}\right|_{\lambda=0} = (-1)^m \Bigg[\sum_{p\prec m, u\prec s\le |p|-1} &\frac{\alpha_p \alpha_{\hat{u}}(|p|+|u|-1)!}{(s+|u|)!(|p|-1-s)!}\cdot(-f)^{-|p|-|u|}\cdot\left(\prod_{i=1}^{s}\left[\frac{d^i f}{dy^i}\right]^{e_{ui}}\right) \\ &\cdot\frac{d^{|p|-1-s}}{dy^{|p|-1-s}}\left(\prod_{i=1}^{m}\left[\frac{d^{i-1}(\mu_{i,c}f)}{dy^{i-1}}\right]^{e_{pi}}\right)\Bigg]_{y=q_\alpha(\tilde{Y})},\end{aligned} \tag{4.212}$$

which is the formula for arbitrary derivatives of VaR. Written without abbreviations this is

$$\begin{aligned}\left.\frac{d^m VaR_\alpha(\tilde{Y}+\lambda\tilde{Z})}{d\lambda^m}\right|_{\lambda=0} = (-1)^m \Bigg[\sum_{p\prec m, u\prec s\le |p|-1} &\frac{\alpha_p \alpha_{\hat{u}}(|p|+|u|-1)!}{(s+|u|)!(|p|-1-s)!}\cdot(-f_Y(y))^{-|p|-|u|} \\ &\cdot\left(\prod_{i=1}^{s}\left[\frac{d^i f_Y(y)}{dy^i}\right]^{e_{ui}}\right)\cdot\frac{d^{|p|-1-s}}{dy^{|p|-1-s}} \\ &\cdot\left(\prod_{i=1}^{m}\left[\frac{d^{i-1}\left(\mathbb{E}(\tilde{Z}^m \mid \tilde{Y}=y)f_Y(y)\right)}{dy^{i-1}}\right]^{e_{pi}}\right)\Bigg]_{y=q_\alpha(\tilde{Y})},\end{aligned} \tag{4.213}$$

with $\alpha_p = \frac{m!}{(1!)^{e_{p1}} e_{p,1}! \cdot \ldots \cdot (m!)^{e_{p,m}} e_{p,m}!}$.

4.5.7 Determination of the First Five Derivatives of VaR

The general form of the mth derivative of VaR is given by (4.213). Subsequently, the first five derivatives will be determined with this formula. For each derivative, we have summands for all partitions $p \prec m$ and $u \prec s \leq |p| - 1$. For the considered cases $1 \leq m \leq 5$, the following partitions $p \prec m$ exist:

$$\begin{aligned} &p \prec 1 = \{1^1\}; \\ &p \prec 2 = \{1^2,\ 2^1\}; \\ &p \prec 3 = \{1^3,\ 1^1 2^1,\ 3^1\}; \\ &p \prec 4 = \{1^4,\ 1^2 2^1,\ 2^2,\ 1^1 3^1,\ 4^1\}; \\ &p \prec 5 = \{1^5,\ 1^3 2^1,\ 1^1 2^2,\ 1^2 3^1,\ 2^1 3^1,\ 1^1 4^1,\ 5^1\}. \end{aligned} \tag{4.214}$$

By construction, the expectation of the unsystematic loss is zero:

$$\mu_{1,c}(y) = \mathbb{E}\left(\tilde{Z}^1 \mid \tilde{Y} = y\right) = 0, \tag{4.215}$$

which is called the "granularity adjustment condition". Consequently, for all partitions with $e_{p1} \neq 0$, the summands of (4.213) are zero, too:

$$\begin{aligned} \prod_{i=1}^{m} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}}\right]^{e_{pi}} &= 0^{e_{p1}} \cdot \prod_{i=2}^{m} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}}\right]^{e_{pi}} \\ &= \begin{cases} \prod_{i=2}^{m} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}}\right]^{e_{pi}} & \text{if } e_{p1} = 0, \\ 0 & \text{if } e_{p1} \neq 0. \end{cases} \end{aligned} \tag{4.216}$$

Hence, the only relevant partitions $p \prec m$ of (4.214) with non-zero terms and the corresponding numbers $|p|$ are given as[260]

$$\begin{aligned} &p \prec 1 = \{1^1\} && \text{with } \left|p = 1^1\right| = 1, \\ &p \prec 2 = \{2^1\} && \text{with } \left|p = 2^1\right| = 1, \\ &p \prec 3 = \{3^1\} && \text{with } \left|p = 3^1\right| = 1, \\ &p \prec 4 = \{4^1,\ 2^2\} && \text{with } \left|p = 4^1\right| = 1,\ \left|p = 2^2\right| = 2, \\ &p \prec 5 = \{5^1,\ 2^1 3^1\} && \text{with } \left|p = 5^1\right| = 1,\ \left|p = 2^1 3^1\right| = 2. \end{aligned} \tag{4.217}$$

For the associated terms

[260] In order to demonstrate that the resulting formula is also valid for $m = 1$, the summand for partition $\{1^1\}$, which equals zero due to argument (4.216), is still considered.

$$\alpha_p = \frac{m!}{(1!)^{e_{p1}} e_{p,1}! \cdot \ldots \cdot (m!)^{e_{p,m}} e_{p,m}!}, \tag{4.218}$$

we obtain

$$\begin{aligned}
\alpha_{1^1} &= \frac{1!}{(1!)^1 \cdot 1!} = 1, \\
\alpha_{2^1} &= \frac{2!}{(2!)^1 \cdot 1!} = 1, \\
\alpha_{3^1} &= \frac{3!}{(3!)^1 \cdot 1!} = 1, \\
\alpha_{4^1} &= \frac{4!}{(4!)^1 \cdot 1!} = 1, \quad \alpha_{2^2} = \frac{4!}{(2!)^2 \cdot 2!} = \frac{24}{8} = 3, \\
\alpha_{5^1} &= \frac{5!}{(5!)^1 \cdot 1!} = 1, \quad \alpha_{2^1 3^1} = \frac{5!}{(2!)^1 \cdot 1! \cdot (3!)^1 \cdot 1!} = \frac{120}{12} = 10.
\end{aligned} \tag{4.219}$$

According to (4.217), we only have $|p| = 1$ and $|p| = 2$, leading to the following partitions $u \prec s \leq |p| - 1$:

$$\begin{aligned}
|p| = 1: &\quad u \prec (s = 0) = \{0\}, \\
|p| = 2: &\quad u \prec \{s = 0,\ s = 1\} = \left\{0,\ 1^1\right\}.
\end{aligned} \tag{4.220}$$

As we have one summand for each $p \prec m$ and $u \prec s \leq (|p| - 1)$, we obtain one summand for $m = 1,\ 2,\ 3$ and three summands for $m = 4,\ 5$:

$$\left.\frac{d^m q}{d\lambda^m}\right|_{\lambda=0} = \begin{cases} (I), & \text{if } m = 1,\ 2,\ 3, \\ (I) + (II) + (III), & \text{if } m = 4,\ 5, \end{cases} \tag{4.221}$$

where the summands are determined with the following variables:

$$\begin{aligned}
&(I) \quad m = 1,\ \ldots,\ 5 : p = m^1, \qquad |p| = 1,\ u \prec (s = 0) = \{0\}, \\
&(II) \quad \left.\begin{aligned} m = 4: &\quad p = 2^2, \\ m = 5: &\quad p = 2^1 3^1, \end{aligned}\right\} \quad |p| = 2,\ u \prec (s = 0) = \{0\}, \\
&(III) \quad \left.\begin{aligned} m = 4: &\quad p = 2^2, \\ m = 5: &\quad p = 2^1 3^1, \end{aligned}\right\} \quad |p| = 2,\ u \prec (s = 1) = \left\{1^1\right\}.
\end{aligned} \tag{4.222}$$

The first summand (*I*), with $p = m^1,\ |p| = 1,\ s = 0,\ u = 0, |u| = 0,\ \hat{u} = 1^1$, $e_{pm} = 1$, and $e_{pi} = 0$ for all $i \neq m$, equals:[261]

[261] For ease of notation, the arguments $\lambda = 0$ of the left-hand as well as $y = q_\alpha(\tilde{Y})$ at the right-hand side are omitted.

$$
\begin{aligned}
(I) &= \frac{\alpha_p \alpha_{\hat{u}}(|p|+|u|-1)!}{(s+|u|)!(|p|-1-s)!}(-f)^{-|p|-|u|} \\
&\quad \cdot \left(\prod_{i=1}^{s}\left[\frac{d^i f}{dy^i}\right]^{e_{ui}}\right) \cdot \frac{d^{|p|-1-s}}{dy^{|p|-1-s}}\left(\prod_{i=1}^{m}\left[\frac{d^{i-1}\left(\mu_{i,c}f\right)}{dy^{i-1}}\right]^{e_{pi}}\right) \\
&= \frac{1\cdot 1\cdot(1+0-1)!}{(0+0)!(1-1-0)!}(-f)^{-1-0}\left(\prod_{i=1}^{0}\left[\frac{d^i f}{dy^i}\right]^{e_{ui}}\right) \cdot \frac{d^{1-1-0}}{dy^{1-1-0}}\left(\prod_{i=1}^{m}\left[\frac{d^{i-1}\left(\mu_{i,c}f\right)}{dy^{i-1}}\right]^{e_{pi}}\right) \\
&= -\frac{1}{f}\cdot\frac{d^{m-1}\left(\mu_{m,c}f\right)}{dy^{m-1}}.
\end{aligned} \tag{4.223}
$$

For $m = 4$, the second summand ($II.[4]$), with values $p = 2^2,\ |p| = 2,\ s = 0,$ $u = 0, |u| = 0,\ \hat{u} = 1^1$, $e_{p2} = 2$, and $e_{pi} = 0$ for all $i \neq 2$, is equivalent to

$$
\begin{aligned}
II.[4] &= \frac{\alpha_p \alpha_{\hat{u}}(|p|+|u|-1)!}{(s+|u|)!(|p|-1-s)!}(-f)^{-|p|-|u|} \\
&\quad \cdot \left(\prod_{i=1}^{s}\left[\frac{d^i f}{dy^i}\right]^{e_{ui}}\right) \cdot \frac{d^{|p|-1-s}}{dy^{|p|-1-s}}\left(\prod_{i=1}^{m}\left[\frac{d^{i-1}\left(\mu_{i,c}f\right)}{dy^{i-1}}\right]^{e_{pi}}\right) \\
&= \frac{3\cdot 1\cdot(2+0-1)!}{(0+0)!(2-1-0)!}(-f)^{-2-0}\left(\prod_{i=1}^{0}\left[\frac{d^i f}{dy^i}\right]^{e_{ui}}\right) \cdot \frac{d^{2-1-0}}{dy^{2-1-0}}\left(\prod_{i=1}^{4}\left[\frac{d^{i-1}\left(\mu_{i,c}f\right)}{dy^{i-1}}\right]^{e_{pi}}\right) \\
&= 3\cdot\frac{1}{f^2}\cdot\frac{d}{dy}\left[\frac{d\left(\mu_{2,c}f\right)}{dy}\right]^2.
\end{aligned} \tag{4.224}
$$

For $m = 5$, we have $p = 2^1 3^1,\ |p| = 2,\ s = 0,\ u = 0, |u| = 0,\ \hat{u} = 1^1$, $e_{p2} = 1,\ e_{p3} = 1,$ and $e_{pi} = 0$ for all $i \neq 2, 3$, leading to

$$
\begin{aligned}
II.[5] &= \frac{\alpha_p \alpha_{\hat{u}}(|p|+|u|-1)!}{(s+|u|)!(|p|-1-s)!}(-f)^{-|p|-|u|}\left(\prod_{i=1}^{s}\left[\frac{d^i f}{dy^i}\right]^{e_{ui}}\right) \\
&\quad \cdot \frac{d^{|p|-1-s}}{dy^{|p|-1-s}}\left(\prod_{i=1}^{m}\left[\frac{d^{i-1}\left(\mu_{i,c}f\right)}{dy^{i-1}}\right]^{e_{pi}}\right) \\
&= \frac{10\cdot 1\cdot(2+0-1)!}{(0+0)!(2-1-0)!}(-f)^{-2-0}\left(\prod_{i=1}^{0}\left[\frac{d^i f}{dy^i}\right]^{e_{uj}}\right) \cdot \frac{d^{2-1-0}}{dy^{2-1-0}}\left(\prod_{i=1}^{5}\left[\frac{d^{i-1}\left(\mu_{i,c}f\right)}{dy^{i-1}}\right]^{e_{pi}}\right) \\
&= 10\cdot\frac{1}{f^2}\cdot\frac{d}{dy}\left(\left[\frac{d\left(\mu_{2,c}f\right)}{dy}\right]\left[\frac{d^2\left(\mu_{3,c}f\right)}{dy^2}\right]\right).
\end{aligned} \tag{4.225}
$$

The third summand for $m = 4$ ($III.[4]$), with $p = 2^2,\ |p| = 2,\ s = 1,\ u = 1^1,$ $|u| = 1,\ \hat{u} = 2^1,\ e_{p2} = 2,\ e_{pi} = 0$ for all $i \neq 2$, and $e_{u1} = 1$ equals

$$\begin{aligned}
III.[4] =& \frac{\alpha_p \alpha_{\hat{u}} (|p| + |u| - 1)!}{(s + |u|)!(|p| - 1 - s)!} (-f)^{-|p|-|u|} \\
&\cdot \left(\prod_{i=1}^{s} \left[\frac{d^i f}{dy^i} \right]^{e_{ui}} \right) \cdot \frac{d^{|p|-1-s}}{dy^{|p|-1-s}} \left(\prod_{i=1}^{m} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}} \right]^{e_{pi}} \right) \\
=& \frac{3 \cdot 1 \cdot (2+1-1)!}{(1+1)!(2-1-1)!} (-f)^{-2-1} \left(\prod_{i=1}^{1} \left[\frac{d^i f}{dy^i} \right]^{1} \right) \cdot \frac{d^{2-1-1}}{dy^{2-1-1}} \left(\prod_{i=1}^{4} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}} \right]^{e_{pi}} \right) \\
=& -3 \cdot \frac{1}{f^3} \cdot \frac{df}{dy} \cdot \left[\frac{d\left(\mu_{2,c} f\right)}{dy} \right]^2 .
\end{aligned} \tag{4.226}$$

For $m = 5$, we have $p = 2^1 3^1$, $|p| = 2$, $s = 1$, $u = 1^1$, $|u| = 1$, $\hat{u} = 2^1$, $e_{p2} = 1$, $e_{p3} = 1,\ e_{pi} = 0$ for all $i \neq 2, 3$, and $e_{u1} = 1$. Hence, we get

$$\begin{aligned}
III.[5] =& \frac{\alpha_p \alpha_{\hat{u}} (|p| + |u| - 1)!}{(s + |u|)!(|p| - 1 - s)!} (-f)^{-|p|-|u|} \\
&\cdot \left(\prod_{i=1}^{s} \left[\frac{d^i f}{dy^i} \right]^{e_{ui}} \right) \cdot \frac{d^{|p|-1-s}}{dy^{|p|-1-s}} \left(\prod_{i=1}^{m} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}} \right]^{e_{pi}} \right) \\
=& \frac{10 \cdot 1 \cdot (2+1-1)!}{(1+1)!(2-1-1)!} (-f)^{-2-1} \left(\prod_{i=1}^{1} \left[\frac{d^i f}{dy^i} \right]^{1} \right) \cdot \frac{d^{2-1-1}}{dy^{2-1-1}} \left(\prod_{i=1}^{5} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}} \right]^{e_{pi}} \right) \\
=& -10 \cdot \frac{1}{f^3} \cdot \frac{df}{dy} \cdot \left[\frac{d\left(\mu_{2,c} f\right)}{dy} \right] \cdot \left[\frac{d^2\left(\mu_{3,c} f\right)}{dy^2} \right] .
\end{aligned} \tag{4.227}$$

Summing up the relevant elements from (4.223) to (4.227) and multiplying by $(-1)^m$ leads to

$$\left.\frac{dq}{d\lambda}\right|_{\lambda=0} = (-1)^1 \cdot \left(-\frac{1}{f} \right) \cdot \frac{d^{1-1}\left(\mu_{1,c} f\right)}{dy^{1-1}} = \mu_{1,c} = 0, \tag{4.228}$$

$$\left.\frac{d^2 q}{d\lambda^2}\right|_{\lambda=0} = (-1)^2 \cdot \left(-\frac{1}{f} \right) \cdot \frac{d^{2-1}\left(\mu_{2,c} f\right)}{dy^{2-1}} = -\frac{1}{f} \cdot \frac{d\left(\mu_{2,c} f\right)}{dy}, \tag{4.229}$$

$$\left.\frac{d^3 q}{d\lambda^3}\right|_{\lambda=0} = (-1)^3 \cdot \left(-\frac{1}{f} \right) \cdot \frac{d^{3-1}\left(\mu_{3,c} f\right)}{dy^{3-1}} = \frac{1}{f} \cdot \frac{d^2\left(\mu_{3,c} f\right)}{dy^2}, \tag{4.230}$$

$$
\begin{aligned}
\left.\frac{d^4 q}{d\lambda^4}\right|_{\lambda=0} &= (-1)^4 \cdot \left[\left(-\frac{1}{f}\right) \cdot \frac{d^{4-1}\left(\mu_{4,c} f\right)}{dy^{4-1}} + 3 \cdot \frac{1}{f^2} \cdot \frac{d}{dy} \left(\frac{d\left(\mu_{2,c} f\right)}{dy} \right)^2 \right. \\
&\quad \left. -3 \cdot \frac{1}{f^3} \cdot \frac{df}{dy} \cdot \left(\frac{d\left(\mu_{2,c} f\right)}{dy} \right)^2 \right] \\
&= \left(-\frac{1}{f}\right) \cdot \left(\frac{d^3\left(\mu_{4,c} f\right)}{dy^3} - 3 \cdot \frac{d}{dy} \left[\frac{1}{f} \left(\frac{d\left(\mu_{2,c} f\right)}{dy} \right)^2 \right] \right),
\end{aligned} \tag{4.231}
$$

and

$$
\begin{aligned}
\left.\frac{d^5 q}{d\lambda^5}\right|_{\lambda=0} &= (-1)^5 \cdot \left[\left(-\frac{1}{f}\right) \cdot \frac{d^{5-1}\left(\mu_{5,c} f\right)}{dy^{5-1}} + 10 \cdot \frac{1}{f^2} \cdot \frac{d}{dy} \left(\left[\frac{d\left(\mu_{2,c} f\right)}{dy} \right] \left[\frac{d^2\left(\mu_{3,c} f\right)}{dy^2} \right] \right) \right. \\
&\quad \left. -10 \cdot \frac{1}{f^3} \cdot \frac{df}{dy} \cdot \left[\frac{d\left(\mu_{2,c} f\right)}{dy} \right] \cdot \left[\frac{d^2\left(\mu_{3,c} f\right)}{dy^2} \right] \right] \\
&= \frac{1}{f} \cdot \left[\frac{d^4\left(\mu_{5,c} f\right)}{dy^4} - 10 \cdot \frac{d}{dy} \left(\frac{1}{f} \cdot \frac{d\left(\mu_{2,c} f\right)}{dy} \, \frac{d^2\left(\mu_{3,c} f\right)}{dy^2} \right) \right].
\end{aligned} \tag{4.232}
$$

Comparing these terms, we find that the derivatives for $m = 1, \ldots, 5$ can be written as

$$
\begin{aligned}
\left.\frac{d^m q}{d\lambda^m}\right|_{\lambda=0} &= (-1)^m \left(-\frac{1}{f}\right) \left[\frac{d^{m-1}\left(\mu_{m,c} f\right)}{dy^{m-1}} - \kappa(m) \right. \\
&\quad \left. \cdot \frac{d}{dy} \left(\frac{1}{f} \cdot \frac{d\left(\mu_{2,c} f\right)}{dy} \, \frac{d^{m-3}\left(\mu_{m-2,c} f\right)}{dy^{m-3}} \right) \right]
\end{aligned} \tag{4.233}
$$

or without abbreviations as

$$
\begin{aligned}
\left.\frac{d^m VaR_\alpha\left(\tilde{Y} + \lambda \tilde{Z}\right)}{d\lambda^m}\right|_{\lambda=0} &= (-1)^m \left(-\frac{1}{f_Y(y)}\right) \left[\frac{d^{m-1}\left(\mu_m\left(\tilde{Z} \mid \tilde{Y} = y\right) f_Y(y)\right)}{dy^{m-1}} \right. \\
&\quad - \kappa(m) \cdot \frac{d}{dy} \left(\frac{1}{f_Y(y)} \cdot \frac{d\left(\mu_2\left(\tilde{Z} \mid \tilde{Y} = y\right) f_Y(y)\right)}{dy} \right. \\
&\quad \left. \left. \cdot \frac{d^{m-3}\left(\mu_{m-2}\left(\tilde{Z} \mid \tilde{Y} = y\right) f_Y(y)\right)}{dy^{m-3}} \right) \right]_{y = q_\alpha\left(\tilde{Y}\right)},
\end{aligned} \tag{4.234}
$$

with $\kappa(1) = \kappa(2) = 0$, $\kappa(3) = 1$, $\kappa(4) = 3$, and $\kappa(5) = 10$, which is the result of Wilde (2003).

4.5.8 Order of the Derivatives of VaR

For any $m \in \mathbb{N}$, the $(m+1)$th element of the Taylor series can be written as[262]

$$\frac{\lambda^m}{m!}\left[\frac{\partial^m VaR_\alpha(\tilde{Y}+\lambda\tilde{Z})}{\partial\lambda^m}\right]_{\lambda=0} = g \circ \left(\frac{\lambda^m}{m!}\sum_{p\prec m}\prod_{i=1}^{m}\left(\mu_i\left[\tilde{Z}\,|\,\tilde{Y}=y\right]\right)^{e_{pi}}\right)\Bigg|_{y=q_\alpha(\tilde{Y})}, \quad (4.235)$$

with g being a function that is independent of the number of credits n. With μ_i as the ith moment about the origin and η_i as the ith moment about the mean, it is possible to write[263]

$$\begin{aligned}
\lambda^m\sum_{p\prec m}\prod_{i=1}^{m}\left(\mu_i\left[\tilde{Z}|\tilde{Y}=y\right]\right)^{e_{pi}}\Big|_{y=q_\alpha(\tilde{Y})} &= \sum_{p\prec m}\prod_{i=1}^{m}\left(\mu_i\left[\lambda\tilde{Z}|\tilde{Y}=y\right]\right)^{e_{pi}}\Big|_{y=q_\alpha(\tilde{Y})}\\
&= \sum_{p\prec m}\prod_{i=1}^{m}\left(\mu_i\left[\tilde{L}-\mathbb{E}\left(\tilde{L}|\tilde{x}\right)|\tilde{x}=x\right]\right)^{e_{pi}}\Big|_{x=q_{1-\alpha}(\tilde{x})}\\
&= \sum_{p\prec m}\prod_{i=1}^{m}\left(\mu_i\left[\left(\tilde{L}|\tilde{x}=x\right)-\mathbb{E}\left(\tilde{L}|\tilde{x}=x\right)\right]\right)^{e_{pi}}\Big|_{x=q_{1-\alpha}(\tilde{x})}\\
&= \sum_{p\prec m}\prod_{i=1}^{m}\left(\eta_i\left[\tilde{L}|\tilde{x}=x\right]\right)^{e_{pi}}\Big|_{x=q_{1-\alpha}(\tilde{x})}\\
&= \sum_{p\prec m}\prod_{i=1}^{m}\left(\eta_i\left[\tilde{L}|\tilde{Y}=y\right]\right)^{e_{pi}}\Big|_{y=q_\alpha(\tilde{Y})} \qquad (4.236)
\end{aligned}$$

for each m. Thus, the derivatives are given as

$$\frac{\lambda^m}{m!}\left[\frac{\partial^m VaR_\alpha(\tilde{Y}+\lambda\tilde{Z})}{\partial\lambda^m}\right]_{\lambda=0} = g \circ \left(\frac{1}{m!}\sum_{p\prec m}\prod_{i=1}^{m}\left(\eta_i\left[\tilde{L}\,|\,\tilde{Y}=y\right]\right)^{e_{pi}}\right)\Bigg|_{y=q_\alpha(\tilde{Y})}. \quad (4.237)$$

---

[262] Cf. (4.213). The notation $g \circ y$ means that a function g is composed with y.

[263] To illustrate that the first identity holds, an example will be demonstrated for m = 5:

$$\begin{aligned}
\lambda\cdot\sum_{p\prec 5}\prod_{i=1}^{5}\left(\mu_i(\tilde{Z})\right)^{e_{pi}} &= \lambda\cdot\Big(\mu_5(\tilde{Z})+\mu_4(\tilde{Z})\cdot\mu_1(\tilde{Z})+\mu_3(\tilde{Z})\cdot\left(\mu_1(\tilde{Z})\right)^2\\
&\quad +\mu_3(\tilde{Z})\cdot\mu_2(\tilde{Z})+\mu_2(\tilde{Z})\cdot\left(\mu_1(\tilde{Z})\right)^3+\mu_2(\tilde{Z})^2\cdot\mu_1(\tilde{Z})+\left(\mu_1(\tilde{Z})\right)^5\Big)\\
&= \mu_5(\lambda\tilde{Z})+\mu_4(\lambda\tilde{Z})\cdot\mu_1(\lambda\tilde{Z})+\mu_3(\lambda\tilde{Z})\cdot\left(\mu_1(\lambda\tilde{Z})\right)^2\\
&\quad +\mu_3(\lambda\tilde{Z})\cdot\mu_2(\lambda\tilde{Z})+\mu_2(\lambda\tilde{Z})\cdot\left(\mu_1(\lambda\tilde{Z})\right)^3+\mu_2(\lambda\tilde{Z})^2\cdot\mu_1(\lambda\tilde{Z})\\
&\quad +\left(\mu_1(\lambda\tilde{Z})\right)^5.
\end{aligned}$$

Furthermore, see (4.9) for the switch between the systematic loss y and the systematic factor x.

Due to[264]

$$\eta_i\big(\tilde{L}\,|\,\tilde{x}=x\big) = \eta_i{}^*(x)\cdot\sum_{j=1}^{n} w_j{}^i \leq \eta_i{}^*(x)\cdot\left(\frac{b}{a}\right)^i\cdot\frac{1}{n^{i-1}} = O\left(\frac{1}{n^{i-1}}\right),$$

with $0 < a \leq EAD_i \leq b$ for all i, and revisiting (4.235) and (4.236), it is straightforward to see that only for $m = 3$ and $m = 4$ there exist terms which are at maximum of order $O(1/n^2)$:

$$\begin{aligned}
&\sum_{p\prec 3}\prod_{i=1}^{3}\left(\eta_i\big[\tilde{L}\,|\,\tilde{Y}=y\big]\right)^{e_{pi}} = \eta_3\big[\tilde{L}\,|\,\tilde{Y}=y\big] = O\left(\frac{1}{n^2}\right),\\
&\sum_{p\prec 4}\prod_{i=1}^{4}\left(\eta_i\big[\tilde{L}\,|\,\tilde{Y}=y\big]\right)^{e_{pi}} = \eta_4\big[\tilde{L}\,|\,\tilde{Y}=y\big] + \left(\eta_2\big[\tilde{L}\,|\,\tilde{Y}=y\big]\right)^2 = O\left(\frac{1}{n^3}\right) + O\left(\frac{1}{n^2}\right).
\end{aligned} \tag{4.238}$$

All terms with higher derivatives of VaR are at least of Order $O(1/n^3)$.

4.5.9 VaR-Based Second-Order Granularity Adjustment for a Normally Distributed Systematic Factor

For convenience, the summands of the second-order granularity add-on Δl_2 will be calculated separately:

$$\begin{aligned}
\Delta l_2 &= \frac{1}{6\varphi}\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\frac{d}{dx}\left[\frac{\eta_{3,c}\varphi}{d\mu_{1,c}/dx}\right]\right)\\
&\quad + \frac{1}{8\varphi}\frac{d}{dx}\left[\frac{1}{\varphi}\frac{1}{d\mu_{1,c}/dx}\left(\frac{d}{dx}\left[\frac{\eta_{2,c}\varphi}{d\mu_{1,c}/dx}\right]\right)^2\right]\Bigg|_{x=\Phi^{-1}(1-\alpha)}\\
&=: \Delta l_{2,1} + \Delta l_{2,2}\big|_{x=\Phi^{-1}(1-\alpha)}.
\end{aligned} \tag{4.239}$$

[264]See (4.14).

The term $\Delta l_{2,1}$ equals

$$\begin{aligned}\Delta l_{2,1} &= \frac{1}{6}\left[\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right)\frac{1}{\varphi}\frac{d}{dx}\left(\frac{\eta_{3,c}\varphi}{d\mu_{1,c}/dx}\right)+\frac{1}{d\mu_{1,c}/dx}\frac{1}{\varphi}\frac{d^2}{dx^2}\left(\frac{\eta_{3,c}\varphi}{d\mu_{1,c}/dx}\right)\right]\\ &= \frac{1}{6}\left[\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right)\left(\underbrace{\frac{1}{\varphi}\frac{d}{dx}\left(\eta_{3,c}\varphi\right)}_{=:A}\frac{1}{d\mu_{1,c}/dx}+\eta_{3,c}\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right)\right)\right.\\ &\quad\left.+\frac{1}{d\mu_{1,c}/dx}\frac{1}{\varphi}\frac{d}{dx}\left[\underbrace{\frac{d}{dx}\left(\eta_{3,c}\varphi\right)\frac{1}{d\mu_{1,c}/dx}}_{=:B}+\underbrace{\eta_{3,c}\varphi\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right)}_{=:C}\right]\right].\end{aligned} \tag{4.240}$$

For the calculation, we need the first and second derivative of the density function φ. As the systematic factor is assumed to be normally distributed, we have

$$\varphi = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \tag{4.241}$$

$$\frac{d\varphi}{dx} = (-x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} = -x\,\varphi, \tag{4.242}$$

$$\frac{d^2\varphi}{dx^2} = (-1)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} - x(-x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} = (x^2-1)\varphi. \tag{4.243}$$

Furthermore, we need the derivative

$$\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right) = -\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}. \tag{4.244}$$

Herewith, the term A form (4.240) can easily be calculated:

$$A = \frac{1}{\varphi}\frac{d}{dx}\left(\eta_{3,c}\varphi\right) = \frac{d\eta_{3,c}}{dx}+\frac{\eta_{3,c}}{\varphi}\frac{d\varphi}{dx} = \frac{d\eta_{3,c}}{dx}-\eta_{3,c}x. \tag{4.245}$$

Furthermore, dB/dx is equal to

$$\begin{aligned}\frac{dB}{dx} &= \frac{d}{dx}\left(\frac{d}{dx}\left(\eta_{3,c}\varphi\right)\frac{1}{d\mu_{1,c}/dx}\right)\\ &= \frac{d^2}{dx^2}\left(\eta_{3,c}\varphi\right)\frac{1}{d\mu_{1,c}/dx} + \frac{d}{dx}\left(\eta_{3,c}\varphi\right)\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right)\\ &= \frac{d}{dx}\left(\frac{d\eta_{3,c}}{dx}\varphi + \eta_{3,c}\frac{d\varphi}{dx}\right)\frac{1}{d\mu_{1,c}/dx} + \left(\frac{d\eta_{3,c}}{dx}\varphi + \eta_{3,c}\frac{d\varphi}{dx}\right)\left(-\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right)\\ &= \left(\frac{d^2\eta_{3,c}}{dx^2}\varphi + 2\frac{d\eta_{3,c}}{dx}\frac{d\varphi}{dx} + \eta_{3,c}\frac{d^2\varphi}{dx^2}\right)\frac{1}{d\mu_{1,c}/dx}\\ &\quad - \left(\frac{d\eta_{3,c}}{dx}\varphi + \eta_{3,c}\frac{d\varphi}{dx}\right)\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}. \end{aligned} \tag{4.246}$$

Similarly, dC/dx is equivalent to

$$\begin{aligned}\frac{dC}{dx} &= \frac{d}{dx}\left(\eta_{3,c}\varphi\left(-\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right)\right)\\ &= -\frac{d}{dx}\left(\eta_{3,c}\varphi\right)\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2} - \eta_{3,c}\varphi\frac{d}{dx}\left(\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right)\\ &= \left(-\frac{d\eta_{3,c}}{dx}\varphi - \eta_{3,c}\frac{d\varphi}{dx}\right)\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\\ &\quad - \eta_{3,c}\varphi\left(\frac{\left(d\mu_{1,c}/dx\right)^2\left(d^3\mu_{1,c}/dx^3\right) - 2\left(d\mu_{1,c}/dx\right)\left(d^2\mu_{1,c}/dx^2\right)^2}{\left(d\mu_{1,c}/dx\right)^4}\right). \end{aligned} \tag{4.247}$$

Using these terms, $\Delta l_{2,1}$ results in

$$\begin{aligned}\Delta l_{2,1} = \frac{1}{6}\Bigg[&-\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\left(\frac{d\eta_{3,c}/dx}{d\mu_{1,c}/dx} - \frac{\eta_{3,c}x}{d\mu_{1,c}/dx} - \eta_{3,c}\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right)\\ &+ \frac{1}{d\mu_{1,c}/dx}\frac{1}{\varphi}\Bigg[\left(\frac{d^2\eta_{3,c}}{dx^2}\varphi + 2\frac{d\eta_{3,c}}{dx}\frac{d\varphi}{dx} + \eta_{3,c}\frac{d^2\varphi}{dx^2}\right)\frac{1}{d\mu_{1,c}/dx}\\ &- 2\left(\frac{d\eta_{3,c}}{dx}\varphi + \eta_{3,c}\frac{d\varphi}{dx}\right)\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\\ &-\eta_{3,c}\varphi\left(\frac{\left(d\mu_{1,c}/dx\right)^2\left(d^3\mu_{1,c}/dx^3\right) - 2\left(d\mu_{1,c}/dx\right)\left(d^2\mu_{1,c}/dx^2\right)^2}{\left(d\mu_{1,c}/dx\right)^4}\right)\Bigg]\Bigg]. \end{aligned} \tag{4.248}$$

Applying the derivatives of φ from (4.242) and (4.243) leads to

$$\begin{aligned}
\Delta l_{2,1} =& \frac{1}{6}\Bigg[-3\frac{\left(d\eta_{3,c}/dx\right)\left(d^2\mu_{1,c}/dx^2\right)}{\left(d\mu_{1,c}/dx\right)^3}+3\frac{\eta_{3,c}x\left(d^2\mu_{1,c}/dx^2\right)}{\left(d\mu_{1,c}/dx\right)^3}+3\eta_{3,c}\frac{\left(d^2\mu_{1,c}/dx^2\right)^2}{\left(d\mu_{1,c}/dx\right)^4} \\
&+\frac{d^2\eta_{3,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}-2x\frac{d\eta_{3,c}/dx}{\left(d\mu_{1,c}/dx\right)^2}+\frac{\eta_{3,c}(x^2-1)}{\left(d\mu_{1,c}/dx\right)^2}-\eta_{3,c}\frac{d^3\mu_{1,c}/dx^3}{\left(d\mu_{1,c}/dx\right)^3}\Bigg] \\
=& \frac{1}{6\left(d\mu_{1,c}/dx\right)^2}\Bigg[\eta_{3,c}\left(x^2-1-\frac{d^3\mu_{1,c}/dx^3}{d\mu_{1,c}/dx}+\frac{3x\left(d^2\mu_{1,c}/dx^2\right)}{d\mu_{1,c}/dx}+\frac{3\left(d^2\mu_{1,c}/dx^2\right)^2}{\left(d\mu_{1,c}/dx\right)^2}\right) \\
&+\frac{d\eta_{3,c}}{dx}\left(-2x-\frac{3\left(d^2\mu_{1,c}/dx^2\right)}{d\mu_{1,c}/dx}\right)+\frac{d^2\eta_{3,c}}{dx^2}\Bigg].
\end{aligned} \tag{4.249}$$

Henceforward, the summand $\Delta l_{2,2}$ will be simplified:

$$\begin{aligned}
\Delta l_{2,2} &= \frac{1}{8\varphi}\frac{d}{dx}\left[\frac{1}{\varphi}\frac{1}{d\mu_{1,c}/dx}\left(\frac{d}{dx}\left[\frac{\eta_{2,c}\varphi}{d\mu_{1,c}/dx}\right]\right)^2\right] \\
&= \frac{1}{8\varphi}\frac{d}{dx}\left(\frac{\varphi}{d\mu_{1,c}/dx}\underbrace{\left(\frac{1}{\varphi}\frac{d}{dx}\left[\frac{\eta_{2,c}\varphi}{d\mu_{1,c}/dx}\right]\right)^2}_{*}\right).
\end{aligned} \tag{4.250}$$

The term (*) is the negative twice of the first-order granularity adjustment, so that we can use the resulting equation (4.18). This leads to

$$\begin{aligned}
\Delta l_{2,2} &= \frac{1}{8\varphi}\frac{d}{dx}\left(\frac{\varphi}{d\mu_{1,c}/dx}\left[-\frac{x\,\eta_{2,c}}{d\mu_{1,c}/dx}+\frac{d\eta_{2,c}/dx}{d\mu_{1,c}/dx}-\frac{\eta_{2,c}d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right]^2\right) \\
&= \frac{1}{8}\Bigg[\underbrace{\frac{1}{\varphi}\frac{d}{dx}\left(\frac{\varphi}{\left(d\mu_{1,c}/dx\right)^3}\right)}_{=:(I)}\left(-x\,\eta_{2,c}+\frac{d\eta_{2,c}}{dx}-\frac{\eta_{2,c}d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)^2 \\
&\quad +\frac{1}{\left(d\mu_{1,c}/dx\right)^3}\underbrace{\frac{d}{dx}\left(\left[-x\,\eta_{2,c}+\frac{d\eta_{2,c}}{dx}-\frac{\eta_{2,c}d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]^2\right)}_{=:(II)}\Bigg].
\end{aligned} \tag{4.251}$$

Using the derivative of a normal distribution $d\varphi/dx = -x\,\varphi$, the term ($I$) is equivalent to

$$
\begin{aligned}
(I) &= \frac{1}{\varphi}\frac{d}{dx}\left(\frac{\varphi}{\left(d\mu_{1,c}/dx\right)^3}\right) \\
&= \frac{1}{\varphi}\frac{d\varphi}{dx}\frac{1}{\left(d\mu_{1,c}/dx\right)^3} + \frac{d}{dx}\left(\frac{1}{\left(d\mu_{1,c}/dx\right)^3}\right) \\
&= \frac{-x}{\left(d\mu_{1,c}/dx\right)^3} - 3\frac{\left(d^2\mu_{1,c}/dx^2\right)}{\left(d\mu_{1,c}/dx\right)^4}.
\end{aligned} \tag{4.252}
$$

Term (II) can be written as

$$
\begin{aligned}
(II) &= \frac{d}{dx}\left(\left[-x\,\eta_{2,c} + \frac{d\eta_{2,c}}{dx} - \frac{\eta_{2,c}d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]^2\right) \\
&= 2\left(-x\,\eta_{2,c} + \frac{d\eta_{2,c}}{dx} - \frac{\eta_{2,c}d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\left(-\eta_{2,c} - x\frac{d\eta_{2,c}}{dx} + \frac{d^2\eta_{2,c}}{dx^2}\right. \\
&\quad \left. -\frac{d}{dx}\left(\eta_{2,c}\frac{d^2\mu_{1,c}}{dx^2}\right)\frac{1}{d\mu_{1,c}/dx} - \eta_{2,c}\frac{d^2\mu_{1,c}}{dx^2}\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right)\right) \\
&= 2\left(-x\eta_{2,c} + \frac{d\eta_{2,c}}{dx} - \frac{\eta_{2,c}d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\left(-\eta_{2,c} - x\frac{d\eta_{2,c}}{dx} + \frac{d^2\eta_{2,c}}{dx^2}\right. \\
&\quad \left. -\frac{d\eta_{2,c}}{dx}\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx} - \eta_{2,c}\frac{d^3\mu_{1,c}/dx^3}{d\mu_{1,c}/dx} + \eta_{2,c}\frac{d^2\mu_{1,c}}{dx^2}\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right).
\end{aligned} \tag{4.253}
$$

Using these expressions, $\Delta l_{2,2}$ from (4.251) is equal to

$$
\begin{aligned}
\Delta l_{2,2} = \frac{1}{8}&\left[\left(\frac{-x}{\left(d\mu_{1,c}/dx\right)^3} - 3\frac{\left(d^2\mu_{1,c}/dx^2\right)}{\left(d\mu_{1,c}/dx\right)^4}\right)\left(-x\,\eta_{2,c} + \frac{d\eta_{2,c}}{dx} - \frac{\eta_{2,c}d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)^2\right. \\
&+ \frac{2}{\left(d\mu_{1,c}/dx\right)^3}\left(-x\,\eta_{2,c} + \frac{d\eta_{2,c}}{dx} - \frac{\eta_{2,c}d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\left(-\eta_{2,c} - x\frac{d\eta_{2,c}}{dx} + \frac{d^2\eta_{2,c}}{dx^2}\right. \\
&\left.\left. -\frac{d\eta_{2,c}}{dx}\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx} - \eta_{2,c}\frac{d^3\mu_{1,c}/dx^3}{d\mu_{1,c}/dx} + \eta_{2,c}\frac{d^2\mu_{1,c}}{dx^2}\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right)\right],
\end{aligned} \tag{4.254}
$$

which leads to

$$\Delta l_{2,2} = \frac{1}{8\left(d\mu_{1,c}/dx\right)^3}\left[\left(-x-3\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\left(\eta_{2,c}\left[-x-\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]+\frac{d\eta_{2,c}}{dx}\right)^2\right.$$
$$+2\left(\eta_{2,c}\left[x+\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]-\frac{d\eta_{2,c}}{dx}\right)\left(\eta_{2,c}\left[1+\frac{d^3\mu_{1,c}/dx^3}{d\mu_{1,c}/dx}-\frac{\left(d^2\mu_{1,c}/dx^2\right)^2}{\left(d\mu_{1,c}/dx\right)^2}\right]\right.$$
$$\left.\left.+\frac{d\eta_{2,c}}{dx}\left[x+\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]-\frac{d^2\eta_{2,c}}{dx^2}\right)\right]. \tag{4.255}$$

Adding the terms $\Delta l_{2,1}$ and $\Delta l_{2,2}$ together results in

$$\Delta l_2 = \frac{1}{6\left(d\mu_{1,c}/dx\right)^2}\left[\eta_{3,c}\left(x^2-1-\frac{d^3\mu_{1,c}/dx^3}{d\mu_{1,c}/dx}+\frac{3x\left(d^2\mu_{1,c}/dx^2\right)}{d\mu_{1,c}/dx}+\frac{3\left(d^2\mu_{1,c}/dx^2\right)^2}{\left(d\mu_{1,c}/dx\right)^2}\right)\right.$$
$$\left.+\frac{d\eta_{3,c}}{dx}\left(-2x-\frac{3\left(d^2\mu_{1,c}/dx^2\right)}{d\mu_{1,c}/dx}\right)+\frac{d^2\eta_{3,c}}{dx^2}\right]$$
$$+\frac{1}{8\left(d\mu_{1,c}/dx\right)^3}\left[\left(-x-3\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\left(\eta_{2,c}\left[-x-\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]+\frac{d\eta_{2,c}}{dx}\right)^2\right.$$
$$+2\left(\eta_{2,c}\left[x+\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]-\frac{d\eta_{2,c}}{dx}\right)\left(\eta_{2,c}\left[1+\frac{d^3\mu_{1,c}/dx^3}{d\mu_{1,c}/dx}-\frac{\left(d^2\mu_{1,c}/dx^2\right)^2}{\left(d\mu_{1,c}/dx\right)^2}\right]\right.$$
$$\left.\left.\left.+\frac{d\eta_{2,c}}{dx}\left[x+\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]-\frac{d^2\eta_{2,c}}{dx^2}\right)\right]\right|_{x=\Phi^{-1}(1-\alpha)}. \tag{4.256}$$

4.5.10 Third Conditional Moment of Losses

Subsequently, the third conditional moment of the portfolios loss about the mean, $\eta_{3,c} = \eta_3(\tilde{L}\,|\,\tilde{x}=x)$, shall be expressed in terms of the moments of separated factors $\widetilde{LGD}_i$ and $1_{\{\tilde{D}_i\}}$. With

$$\begin{aligned}\eta_{3,c} &= \eta_3\left(\tilde{L}\,|\,\tilde{x}=x\right)\\ &= \eta_3\left(\sum_{i=1}^{n} w_i\cdot\widetilde{LGD}_i\cdot 1_{\{\tilde{D}_i\}}\,|\,\tilde{x}=x\right)\\ &= \sum_{i=1}^{n} w_i^{\,3}\cdot\eta_3\left(\widetilde{LGD}_i\cdot 1_{\{\tilde{D}_i\}}\,|\,\tilde{x}=x\right),\end{aligned} \tag{4.257}$$

which is due to the conditional independence property, we need to determine $\eta_3(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}\,|\,\tilde{x})$. In general, the third moment about the mean is equal to

$$\begin{aligned}\eta_3(\tilde{X}) &= \mathbb{E}\left(\left[\tilde{X}-\mathbb{E}(\tilde{X})\right]^3\right)\\ &= \mathbb{E}\left[\tilde{X}^3 - 3\tilde{X}^2\mathbb{E}(\tilde{X}) + 3\tilde{X}\mathbb{E}^2(\tilde{X}) - \mathbb{E}^3(\tilde{X})\right]\\ &= \mathbb{E}\left(\tilde{X}^3\right) - 3\mathbb{E}\left(\tilde{X}^2\right)\mathbb{E}(\tilde{X}) + 3\mathbb{E}(\tilde{X})\mathbb{E}^2(\tilde{X}) - \mathbb{E}^3(\tilde{X})\\ &= \mathbb{E}\left(\tilde{X}^3\right) - 3\mathbb{E}\left(\tilde{X}^2\right)\mathbb{E}(\tilde{X}) + 2\mathbb{E}^3(\tilde{X}).\end{aligned} \tag{4.258}$$

Thus, the conditional moment $\eta_3(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}\,|\,\tilde{x})$ can be written as

$$\begin{aligned}\eta_3\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}\,|\,\tilde{x}\right) &= \mathbb{E}\left(\left[\widetilde{LGD} \cdot 1_{\{\tilde{D}_i\}}\,|\,\tilde{x}\right]^3\right)\\ &\quad - 3\,\mathbb{E}\left(\left[\widetilde{LGD} \cdot 1_{\{\tilde{D}_i\}}\,|\,\tilde{x}\right]^2\right) \cdot \mathbb{E}\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}\,|\,\tilde{x}\right)\\ &\quad + 2\,\mathbb{E}^3\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}\,|\,\tilde{x}\right).\end{aligned} \tag{4.259}$$

Using the conditional independence property again, considering that the LGDs are assumed to be stochastically independent of each other, and with $\mathbb{E}[(1_{\{\tilde{D}_i\}}\,|\,\tilde{x})^i] = \mathbb{E}[(1_{\{\tilde{D}_i\}}\,|\,\tilde{x})] = p(\tilde{x})$, we have

$$\begin{aligned}\eta_3\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}\,|\,\tilde{x}\right) &= \mathbb{E}\left(\left[\widetilde{LGD}\,|\,\tilde{x}\right]^3\right)p(\tilde{x}) - 3\,\mathbb{E}\left(\left[\widetilde{LGD}\,|\,\tilde{x}\right]^2\right)\mathbb{E}\left(\widetilde{LGD}\,|\,\tilde{x}\right)p^2(\tilde{x})\\ &\quad + 2\,\mathbb{E}^3\left(\widetilde{LGD}\,|\,\tilde{x}\right)p^3(\tilde{x})\\ &= \mathbb{E}\left(\widetilde{LGD}^3\right)p(\tilde{x}) - 3\,\mathbb{E}\left(\widetilde{LGD}^2\right)\mathbb{E}\left(\widetilde{LGD}\right)p^2(\tilde{x})\\ &\quad + 2\,\mathbb{E}^3\left(\widetilde{LGD}\right)p^3(\tilde{x}).\end{aligned} \tag{4.260}$$

With the abbreviations $ELGD = \mathbb{E}(\widetilde{LGD})$, $VLGD = \mathbb{V}(\widetilde{LGD})$ as well as $SLGD = \eta_3(\widetilde{LGD})$ and using (4.258) again, we obtain

$$\mathbb{E}\left(\widetilde{LGD}^2\right) = ELGD^2 + VLGD, \tag{4.261}$$

$$\begin{aligned}\mathbb{E}\left(\widetilde{LGD}^3\right) &= SLGD + 3(ELGD^2 + VLGD)ELGD - 2\,ELGD^3\\ &= ELGD^3 + 3\,ELGD \cdot VLGD + SLGD.\end{aligned} \tag{4.262}$$

Consequently, (4.260) is equivalent to

$$\eta_3\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} \,|\, \tilde{x}\right) = \left(ELGD^3 + 3\,ELGD \cdot VLGD + SLGD\right)p(\tilde{x}) - 3\left(ELGD^3 + ELGD \cdot VLGD\right)p^2(\tilde{x}) + 2\,ELGD^3 p^3(\tilde{x}). \tag{4.263}$$

Thus, the conditional moment of the portfolio loss (4.257) can finally be written as

$$\begin{aligned}\eta_{3,c} &= \sum_{i=1}^{n} {w_i}^3 \cdot \eta_3\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} \,|\, \tilde{x} = x\right) \\ &= \sum_{i=1}^{n} {w_i}^3 \left[\left({ELGD_i}^3 + 3 \cdot ELGD_i \cdot VLGD_i + SLGD_i\right) \cdot p_i(x)\right. \\ &\quad \left. -3 \cdot \left({ELGD_i}^3 + ELGD_i \cdot VLGD_i\right) \cdot {p_i}^2(x) + 2 \cdot {ELGD_i}^3 \cdot {p_i}^3(x)\right]. \end{aligned} \tag{4.264}$$

4.5.11 Difference Between the VaR Definitions

For the case of homogeneous credits and with $LGD = 1$, the possible realizations of losses are

$$l \in \left\{0,\ \frac{1}{n},\ \frac{2}{n},\ \ldots,\ \frac{n-1}{n},\ 1\right\}, \tag{4.265}$$

which implies

$$\mathbb{P}\left[\tilde{L} \le l\right] = \mathbb{P}\left[\tilde{L} < (l + 1/n)\right]. \tag{4.266}$$

If we define $l_2 := l_1 + 1/n$, we get

$$\begin{aligned} VaR_\alpha^{(-)}\left(\tilde{L}\right) &= \sup\left\{l_1 \in \mathbb{R} \,|\, \mathbb{P}\left[\tilde{L} \le l_1\right] < \alpha\right\} \\ &= \sup\left\{l_1 \in \mathbb{R} \,|\, \mathbb{P}\left[\tilde{L} < \left(l_1 + \frac{1}{n}\right)\right] < \alpha\right\} \\ &= \sup\left\{\left(l_2 - \frac{1}{n}\right) \in \mathbb{R} \,|\, \mathbb{P}\left[\tilde{L} < l_2\right] < \alpha\right\} \\ &= \sup\left\{l_2 \in \mathbb{R} \,|\, \mathbb{P}\left[\tilde{L} < l_2\right] < \alpha\right\} - \frac{1}{n} \\ &= VaR_\alpha^{(+)}\left(\tilde{L}\right) - \frac{1}{n}. \end{aligned} \tag{4.267}$$

4.5.12 Identity of ES Within the Basel Framework

Using the result of the ASRF framework (2.93), the definition of the ES (2.19), the integral representation of the conditional expectation, and the identity of the condition as in (4.9), the ES of the portfolio loss equals

$$\begin{aligned}
ES_\alpha^{(\text{Basel})}\left(\tilde{L}\right) &= ES_\alpha\left[\mathbb{E}\left(\tilde{L}\,|\,\tilde{x}\right)\right] \\
&= ES_\alpha\left[\mu_{1,c}(\tilde{x})\right] \\
&= \frac{1}{1-\alpha}\left[\mathbb{E}\left(\mu_{1,c}(\tilde{x})\,|\,\mu_{1,c}(\tilde{x}) \geq q_\alpha\left(\mu_{1,c}(\tilde{x})\right)\right)\right] \\
&= \frac{1}{1-\alpha}\left[\mathbb{E}\left(\mu_{1,c}(\tilde{x})\,|\,\tilde{x} \leq \Phi^{-1}(1-\alpha)\right)\right] \\
&= \frac{1}{1-\alpha}\int\limits_{-\infty}^{\Phi^{-1}(1-\alpha)} \mu_{1,c}(x)\varphi(x)dx.
\end{aligned} \tag{4.268}$$

With the conditional independence property as in (2.92), the conditional PD of the Vasicek model (2.66), the integral representation (2.126), and the symmetry of the normal distribution, the ES can be written as

$$\begin{aligned}
ES_\alpha^{(\text{Basel})}\left(\tilde{L}\right) &= \frac{1}{1-\alpha}\int\limits_{-\infty}^{\Phi^{-1}(1-\alpha)} \sum_{i=1}^{n} \mathbb{E}\left(w_i \cdot \widetilde{LGD}_i \cdot 1_{\{D_i\}}\,|\,x\right)\varphi(x)dx \\
&= \frac{1}{1-\alpha}\sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \int\limits_{-\infty}^{\Phi^{-1}(1-\alpha)} p_i(x)\varphi(x)dx \\
&= \frac{1}{1-\alpha}\sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \int\limits_{-\infty}^{\Phi^{-1}(1-\alpha)} \Phi\left(\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i}\cdot x}{\sqrt{1-\rho_i}}\right)\varphi(x)dx \\
&= \frac{1}{1-\alpha}\sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \Phi_2\left(\Phi^{-1}(1-\alpha), \Phi^{-1}(PD_i), \sqrt{\rho_i}\right) \\
&= \frac{1}{1-\alpha}\sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \Phi_2\left(-\Phi^{-1}(\alpha), \Phi^{-1}(PD_i), \sqrt{\rho_i}\right).
\end{aligned} \tag{4.269}$$

4.5.13 Arbitrary Derivatives of ES

According to (2.20), the ES can be written as

$$ES_\alpha(\tilde{L}) = \frac{1}{1-\alpha}\int_\alpha^1 q^u(\tilde{L})du. \tag{4.270}$$

Thus, for continuous distributions, all derivatives of ES can be expressed as

$$\frac{d^m ES_\alpha}{d\lambda^m} = \frac{d^m}{d\lambda^m}\left(\frac{1}{1-\alpha}\int_\alpha^1 q_u du\right) = \frac{1}{1-\alpha}\int_\alpha^1 \frac{d^m q_u}{d\lambda^m}du. \tag{4.271}$$

The derivative of VaR is a function of $f_Y(y)$ and $\mu_{i,c}(y)$ evaluated at $q_u(\tilde{Y})$. The substitution $u = F_Y(y)$, so that $du/dy = f_Y(y)$, $y(u=\alpha) = F_Y^{-1}(\alpha) = q_\alpha(\tilde{Y})$, and $y(u=1) = F_Y^{-1}(1) = \infty$, leads to:[265]

$$\left.\frac{d^m ES_\alpha}{d\lambda^m}\right|_{\lambda=0} = \frac{1}{1-\alpha}\int_{u=\alpha}^1 \left.\frac{d^m q_u}{d\lambda^m}\right|_{\lambda=0} du = \frac{1}{1-\alpha}\int_{y=q_\alpha(\tilde{Y})}^\infty \left.\frac{d^m q_u}{d\lambda^m}\right|_{\lambda=0} f_Y dy, \tag{4.272}$$

where the expression resulting from the derivative of VaR simply has to be evaluated at y since $q_u(\tilde{Y}) = y$. Using the derivatives of VaR from (4.212), this leads to

$$\begin{aligned}\left.\frac{d^m ES_\alpha}{d\lambda^m}\right|_{\lambda=0} =& \frac{1}{1-\alpha}\int_{y=q_\alpha(\tilde{Y})}^\infty (-1)^m \left[\sum_{p\prec m, u\prec s\leq|p|-1} \frac{\alpha_p \alpha_{\hat{u}}(|p|+|u|-1)!}{(s+|u|)!(|p|-1-s)!}\right. \\ &\left.\cdot(-f)^{-|p|-|u|}\cdot\left(\prod_{i=1}^s\left[\frac{d^i f}{dy^i}\right]^{e_{ui}}\right)\cdot\frac{d^{|p|-1-s}}{dy^{|p|-1-s}}\left(\prod_{i=1}^m\left[\frac{d^{i-1}(\mu_{i,c}f)}{dy^{i-1}}\right]^{e_{pi}}\right)\right] f\,dy,\end{aligned} \tag{4.273}$$

with $\alpha_p = \dfrac{m!}{(1!)^{e_{p1}} e_{p,1}! \cdot \ldots \cdot (m!)^{e_{p,m}} e_{p,m}!}$.

[265] Cf. Wilde (2003), p. 11.

4.5.14 Determination of the First Five Derivatives of ES

Instead of solving the integral (4.272) for each of the derivatives of VaR (4.228)–(4.232), we will directly evaluate the integral for the first five derivatives. Using the expression for the first five derivatives of VaR (4.233), we obtain

$$\begin{aligned}\left.\frac{d^m ES}{d\lambda^m}\right|_{\lambda=0} &= \frac{1}{1-\alpha}\int\limits_{y=q_\alpha(\tilde{Y})}^{\infty}\frac{d^m q}{d\lambda^m} f_Y dy\\ &= \frac{1}{1-\alpha}\int\limits_{y=q_\alpha(\tilde{Y})}^{\infty}(-1)^m\left(-\frac{1}{f}\right)\left[\frac{d^{m-1}\left(\mu_{m,c}f\right)}{dy^{m-1}}\right.\\ &\quad\left. - \kappa(m)\cdot\frac{d}{dy}\left(\frac{1}{f}\cdot\frac{d\left(\mu_{2,c}f\right)}{dy}\frac{d^{m-3}\left(\mu_{m-2,c}f\right)}{dy^{m-3}}\right)\right]f\,dy. \end{aligned} \tag{4.274}$$

This term is equal to

$$\begin{aligned}\left.\frac{d^m ES}{d\lambda^m}\right|_{\lambda=0} &= (-1)^m\cdot\frac{1}{1-\alpha}\cdot\left(\int\limits_{y=q_\alpha(\tilde{Y})}^{\infty}\left(-\frac{d^{m-1}\left(\mu_{m,c}f\right)}{dy^{m-1}}\right)dy\right.\\ &\quad\left. + \kappa(m)\cdot\int\limits_{y=q_\alpha(\tilde{Y})}^{\infty}\frac{d}{dy}\left(\frac{1}{f}\cdot\frac{d\left(\mu_{2,c}f\right)}{dy}\cdot\frac{d^{m-3}\left(\mu_{m-2,c}f\right)}{dy^{m-3}}\right)dy\right)\\ &= (-1)^m\cdot\frac{1}{1-\alpha}\cdot\left(\left[-\frac{d^{m-2}\left(\mu_{m,c}f\right)}{dy^{m-2}}\right]_{q_\alpha(\tilde{Y})}^{\infty}\right.\\ &\quad\left. +\kappa(m)\cdot\left[\frac{1}{f}\cdot\frac{d\left(\mu_{2,c}f\right)}{dy}\cdot\frac{d^{m-3}\left(\mu_{m-2,c}f\right)}{dy^{m-3}}\right]_{y=q_\alpha(\tilde{Y})}^{\infty}\right)\\ &= (-1)^m\cdot\frac{1}{1-\alpha}\cdot\left(\frac{d^{m-2}\left(\mu_{m,c}f\right)}{dy^{m-2}}\right.\\ &\quad\left.\left. -\kappa(m)\cdot\left[\frac{1}{f}\cdot\frac{d\left(\mu_{2,c}f\right)}{dy}\cdot\frac{d^{m-3}\left(\mu_{m-2,c}f\right)}{dy^{m-3}}\right]\right)\right|_{y=q_\alpha(\tilde{Y})}, \end{aligned} \tag{4.275}$$

or written without abbreviations as

$$\left.\frac{d^m ES_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}{d\lambda^m}\right|_{\lambda=0} = (-1)^m \cdot \frac{1}{1-\alpha}\cdot\left(\frac{d^{m-2}\left(\mu_m\left(\tilde{Z}\,|\,\tilde{Y}=y\right)f_Y(y)\right)}{dy^{m-2}}\right.$$
$$\left.\left. - \kappa(m)\cdot\left[\frac{1}{f_Y(y)}\cdot\frac{d\left(\mu_2\left(\tilde{Z}\,|\,\tilde{Y}=y\right)f_Y(y)\right)}{dy}\cdot\frac{d^{m-3}\left(\mu_{m-2}\left(\tilde{Z}\,|\,\tilde{Y}=y\right)f_Y(y)\right)}{dy^{m-3}}\right]\right)\right|_{y=q_\alpha\left(\tilde{Y}\right)}, \tag{4.276}$$

with $\kappa(1)=\kappa(2)=0,\ \kappa(3)=1,\ \kappa(4)=3$, and $\kappa(5)=10$. This is the result of Wilde (2003), except that the algebraic signs of Wilde (2003) seem to be wrong.

4.5.15 *ES-Based Second-Order Granularity Adjustment for a Normally Distributed Systematic Factor*

The summands of the second-order granularity add-on Δl_2 can be expressed as

$$\begin{aligned}\Delta l_2 &= \frac{1}{6(1-\alpha)}\frac{1}{d\mu_{1,c}/dx}\frac{d}{dx}\left(\frac{\eta_{3,c}\varphi}{d\mu_{1,c}/dx}\right)\\ &\quad\left.+\frac{1}{8(1-\alpha)}\frac{1}{\varphi}\frac{1}{d\mu_{1,c}/dx}\left[\frac{d}{dx}\left(\frac{\eta_{2,c}\varphi}{d\mu_{1,c}/dx}\right)\right]^2\right|_{x=\Phi^{-1}(1-\alpha)}\\ &=: \Delta l_{2,1}+\Delta l_{2,2}\big|_{x=\Phi^{-1}(1-\alpha)}.\end{aligned}\tag{4.277}$$

Using the derivative of the normal distribution (4.242), the summand $\Delta l_{2,1}$ equals

$$\begin{aligned}\Delta l_{2,1} &= \frac{1}{6(1-\alpha)}\frac{1}{d\mu_{1,c}/dx}\frac{d}{dx}\left(\frac{\eta_{3,c}\varphi}{d\mu_{1,c}/dx}\right)\\ &= \frac{1}{6(1-\alpha)}\frac{1}{d\mu_{1,c}/dx}\left[\frac{d}{dx}\left(\eta_{3,c}\varphi\right)\frac{1}{d\mu_{1,c}/dx}+\eta_{3,c}\varphi\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right)\right]\\ &= \frac{1}{6(1-\alpha)}\frac{1}{d\mu_{1,c}/dx}\left[\left(\frac{d\eta_{3,c}}{dx}\varphi+\eta_{3,c}\frac{d\varphi}{dx}\right)\frac{1}{d\mu_{1,c}/dx}-\eta_{3,c}\varphi\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right]\\ &= \frac{1}{6(1-\alpha)}\frac{\varphi}{\left(d\mu_{1,c}/dx\right)^2}\left[\frac{d\eta_{3,c}}{dx}-\eta_{3,c}\left(x-\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\right].\end{aligned}\tag{4.278}$$

Using the same transformations, the summand $\Delta l_{2,2}$ is equivalent to

$$\begin{aligned}\Delta l_{2,2} &= \frac{1}{8(1-\alpha)}\frac{1}{\varphi}\frac{1}{d\mu_{1,c}/dx}\left[\frac{d}{dx}\left(\frac{\eta_{2,c}\varphi}{d\mu_{1,c}/dx}\right)\right]^2\\ &= \frac{1}{8(1-\alpha)}\frac{1}{\varphi}\frac{1}{d\mu_{1,c}/dx}\left[\frac{1}{d\mu_{1,c}/dx}\left[\frac{d\eta_{2,c}}{dx}\varphi-\eta_{2,c}\varphi\left(x-\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\right]\right]^2\\ &= \frac{1}{8(1-\alpha)}\frac{\varphi}{\left(d\mu_{1,c}/dx\right)^3}\left[\frac{d\eta_{2,c}}{dx}-\eta_{2,c}\left(x-\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\right]^2, \end{aligned}\tag{4.279}$$

leading to a second-order adjustment of

$$\begin{aligned}\Delta l_2 = &\frac{1}{6(1-\alpha)}\frac{\varphi}{\left(d\mu_{1,c}/dx\right)^2}\left[\frac{d\eta_{3,c}}{dx}-\eta_{3,c}\left(x-\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\right]\\ &+\left.\frac{1}{8(1-\alpha)}\frac{\varphi}{\left(d\mu_{1,c}/dx\right)^3}\left[\frac{d\eta_{2,c}}{dx}-\eta_{2,c}\left(x-\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\right]^2\right|_{x=\Phi^{-1}(1-\alpha)}. \end{aligned}\tag{4.280}$$

4.5.16 *Probability Density Function of the Logit-Normal Distribution*

The derivation of the density function is based on the inverse function theorem[266]

$$f_Y(y)=f_X\big(g^{-1}(y)\big)\cdot\left|\frac{dg^{-1}(y)}{dy}\right|. \tag{4.281}$$

For the logit function $\tilde{Y}=e^{\tilde{X}}\big/(1+e^{\tilde{X}})$, we have

$$\begin{aligned} g(x) = y &= \frac{e^x}{1+e^x}=\frac{1}{e^{-x}+1}\\ \Leftrightarrow e^{-x} &= \frac{1}{y}-1\\ \Leftrightarrow g^{-1}(y)=x &= -\ln\left(\frac{1}{y}-1\right)\end{aligned}\tag{4.282}$$

[266] Cf. Appendix 4.5.3.

and

$$\frac{dg^{-1}(y)}{dy} = \frac{d}{dy}\left(-\ln\left(\frac{1}{y}-1\right)\right) = -\frac{1}{\frac{1}{y}-1}\cdot\left(-\frac{1}{y^2}\right) = \frac{1}{y(1-y)}. \tag{4.283}$$

Using the density of a normal distribution (4.82) for f_X and recognizing that y is bounded in the interval [0, 1], we get

$$\begin{aligned} f_Y(y) &= f_X\left(-\ln\left(\frac{1}{y}-1\right)\right)\cdot\left|\frac{1}{y(1-y)}\right| \\ &= \frac{1}{\sqrt{2\pi\sigma_X^2}}\exp\left(-\frac{(-\ln(1/y-1)-\mu_X)^2}{2\sigma_X^2}\right)\cdot\frac{1}{y(1-y)} \\ &= \frac{1}{\sqrt{2\pi\sigma_X^2}}\exp\left(-\frac{(\ln(1/y-1)+\mu_X)^2}{2\sigma_X^2}\right)\cdot\frac{1}{y(1-y)}. \end{aligned} \tag{4.284}$$